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The Calogero-Moser Model Based On Doubly-Laced Lie Algebras

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Abstract

The Calogero-Moser model is an one-dimensional dynamical system that describes N pairwise interacting particles on a line with nonlinear interaction potentials. These potentials are associated with the root system of the Simple Lie Algebras. The Calogero-Moser model is integrable, and its integrability is describe through the Lax pair operators built in the root system of the associated Lie algebra. In the present work, a new Lax pair operator for the Calogero-Moser model based on the Doubly Simply-Laced Lie algebras is presented. It is shown that the canonical equation of motion obtained from the Lax pair formulation and from the Hamiltonian formulation are consistent.

Keywords: *Calogero-Moser Model, Integrability, Doubly-Laced Lie Algebras*

1. Introduction

The Calogero-Moser model,[1] was first independently studied back in the 70s. This model is a dynamical model that describe N particles on a line identified by their coordinates $x_i, i = 1, 2, \ldots, N$ interacting with a pairwise potential $f(x_i, x_j)$. Several realization of this potential have been studied in the literature (for a comprehensive review see ref[2]). It was well known that these potentials, classified into four classes : $1/q^2$-type, $1/\sin^2(q)$-type, $1/\sinh^2(q)$-type and the elliptic type $\wp(q)$ with q is the distance of the interacting pair of particles, describe an integrable theory.

In general this integrability is guaranteed by the Lax pair operator formalism. Recently, Bordner et.al[3] propose a general Lax pair operator for the Calogero-Moser model based on *Simply-Laced* Lie algebras. They have shown that this the new Lax pair operator is consistent in which it reproduces the equation of motion of the Calogero-Moser model in question.

In the present work, a *new* Lax pair operator for the Calogero-Moser model for the first three potential types based on the *Non Simply-Laced* Lie algebra is presented. It will also be shown that the equation of motion derived from this Lax formalism is consistent with the Hamiltonian formalism. It is believed that this Lax pair operator have never been presented in any literature.

2. The Calogero-Moser Model

The Calogero-Moser model is describe by the following Hamiltonian:[3]

$$H = \frac{1}{2}p^2 - \frac{g^2}{2} \sum_{\alpha\in\Delta} x(\alpha \cdot q)x(-\alpha \cdot q), \qquad (1)$$

where q and p are the dynamical variables of the system represented as vectors in $\mathbb{R}^N$, $q = (q^1, q^2, \ldots, q^N) \in \mathbb{R}^N$ and $p = (p^1, p^2, \ldots, p^N) \in \mathbb{R}^N$. Δ is the set of all roots associated with the underlying Lie algebra which are also represented as a vector in $\mathbb{R}^N$,

$$\Delta = \{\alpha, \beta, \gamma, \ldots\}, \qquad \alpha \in \mathbb{R}^N. \qquad (2)$$

We follow the standard notation of the Lie algebras and its representation in root space, see for example the book by Georgi or the book by Varadarajan.[4] The interaction coupling parameter between the particles is designated as g. And the function $x(\alpha \cdot q)$ are unique for each class of Calogero-Moser model. In the following, the function $x(\alpha \cdot q)$ and some related functions are defined for the three class of potential considered in this work.

1. rational potential: $1/q^2$

$$x(t) = \frac{1}{t}, \qquad y(t) = z(t) = -\frac{1}{t^2}. \qquad (3)$$

2. trigonometric potential: $1/\sin^2(q)$

$$x(t) = a\cot(at), \qquad y(t) = z(t) = -\frac{a^2}{\sin^2(at)}. \tag{4}$$

3. hyperbolic potential: $1/\sinh^2(q)$

$$x(t) = a\coth(at), \qquad y(t) = z(t) = -\frac{a^2}{\sinh^2(at)}. \tag{5}$$

Note that the functions $y(t)$ and $z(t)$ are the derivatives of $x(t)$ and these functions satisfy the following *sum rule*

$$y(u)x(v) - y(v)x(u) = x(u+v)[z(u) - z(v)], \tag{6}$$

where $u, v \in \mathbb{C}$.

The canonical equation of motion of the Calogero-Moser model are derived from the above Hamiltonian as follows :

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} = p, & (7)\\ \dot{p} &= -\frac{\partial H}{\partial q} = -\frac{g^2}{2}\sum_{\alpha\in\Delta}[x(\alpha\cdot q)y(-\alpha\cdot q) \\ &\qquad -x(-\alpha\cdot q)y(\alpha\cdot q)]\alpha. & (8)\end{aligned}$$

Note that from the definition of the Hamiltonian, equation (1) above, there exist only *one* coupling parameter in the theory.

3. The Lax Pair Operator

The standard Lax pair operator formalism for integrable system have been widely used not only in linear systems but most importantly in nonlinear cases. In this formalism, an exact solution to the original problem is found from the inverse scattering transform from an associated scattering problem.

Given a system of interacting particles describe by the following Hamiltonian,

$$H = \frac{1}{2}p^2 + u(x,t), \tag{9}$$

with $u(x,t)$ is the interaction potential (in the case considered is a nonlinear interaction potential). The general nonlinear equation of motion derived from this Hamiltonian is given as,

$$u_t = N(u), \tag{10}$$

where $N(u)$ is some nonlinear operator.

It is assumed that the potential $u(x,t)$ is associated with a Hermitean scattering problem operator $L(u) = -\partial_{xx} + u(x,t)$ whose eigenvalue equation is given as,[5)]

$$L\psi = \lambda\psi, \tag{11}$$

with the eigenvalue λ is assumed constant with time. The evolution of the eigenfunction ψ is governed by the operator M,

$$\psi_t = M\psi. \tag{12}$$

From the above, we can deduce that the evolution of the operator $L(u)$ is given as the commutation of the *Lax pair operators* M and L,

$$L_t = u_t \equiv [M, L]. \tag{13}$$

A wisdom most commonly abide, is that a system is said to be *integrable* if there exist a Lax pair operators associated with it. Therefore the integrability of the Calogero-Moser model based on the non simply-laced Lie algebras is claimed by the existence of a Lax pair operators associated to the Calogero-Moser model in question.

The proposed Lax pair operator for the doubly-laced case of the Lie algebras B_n, C_n and F_4 are given as follows,

$$\begin{aligned} L(q,p) &= p\cdot H + X + X_r, & (14)\\ M(q) &= D + Y + Y_r, & (15)\end{aligned}$$

the structure of above definition of the Lax operators are the same as given in ref.[3)] In the above,

$$\begin{aligned} H_{\beta\gamma} &= \beta\,\delta_{\beta\gamma} & (16)\\ D_{\beta\gamma} &= -ig\,[z(\beta\cdot q) + \sum_{\kappa\in\Delta,\ \kappa\cdot\beta=1} z(\kappa\cdot q)], & (17)\end{aligned}$$

and the operators X, X_r, Y and Y_r are redefined as

$$\begin{aligned} X &= i\textstyle\sum_{\alpha\in\Delta} x(\alpha\cdot q)E(\alpha), \\ Y &= i\textstyle\sum_{\alpha\in\Delta} y(\alpha\cdot q)E(\alpha), \end{aligned} \tag{18}$$

$$\begin{aligned} X_r &= i\textstyle\sum_{\alpha\in\Delta} x(\alpha\cdot q)E_d(\alpha), \\ Y_r &= i\textstyle\sum_{\alpha\in\Delta} y(\alpha\cdot q)E_{d'}(\alpha), \end{aligned} \tag{19}$$

with the matrices $E(\alpha), E_d(\alpha)$ and $E_{d'}(\alpha)$ are given as,

$$\begin{aligned} E(\alpha)_{\beta\gamma} &= \delta_{\beta-\gamma,\alpha}(\,g\,\delta_{\alpha^2,2} + g'\,\delta_{\alpha^2,4}), & (20)\\ E_d(\alpha)_{\beta\gamma} &= \delta_{\beta-\gamma,2\alpha}\{\,g\,(2\,\delta_{\beta^2,2}\,\delta_{\gamma^2,2}\,\delta_{\alpha^2,2} \\ &\quad + \delta_{\beta^2,4}\,\delta_{\gamma^2,4}\,\delta_{\alpha^2,2}) \\ &\quad + 2\,g'\,\delta_{\beta^2,4}\,\delta_{\gamma^2,4}\,\delta_{\alpha^2,4}\}, & (21)\\ E_{d'}(\alpha)_{\beta\gamma} &= \delta_{\beta-\gamma,2\alpha}\{\,g\,(\delta_{\beta^2,2}\,\delta_{\gamma^2,2}\,\delta_{\alpha^2,2} \\ &\quad + \delta_{\beta^2,4}\,\delta_{\gamma^2,4}\,\delta_{\alpha^2,2}) \\ &\quad + g'\,\delta_{\beta^2,4}\,\delta_{\gamma^2,4}\,\delta_{\alpha^2,4}\}. & (22)\end{aligned}$$

In the above definition, the operators E_d and $E_{d'}$ are called the double root discriminators. Although in the above definition we have assigned two values of coupling parameters, g which is associated with the *short* root, $\alpha^2 = 2$, and g' which is associated with the *long* root, $\alpha^2 = 4$, it turns out that consistency condition yield a relation between these two coupling parameters, $g' = g\sqrt{2}$.

4. Consistency of the Equation of Motion

In terms of the matrix operators defined in the previous section, the Lax equation (13) is given as,

$$\frac{d(X+X_r)}{dt} = [p\cdot H, Y+Y_r], \qquad (23)$$

$$\frac{dp}{dt}\cdot H = [X+X_r, Y+Y_r]_{diag}, \qquad (24)$$

$$0 = [X+X_r, D+Y+Y_r]_{off\ diag}. \qquad (25)$$

The consistency of the proposed Lax operator is obtained by comparing the Calogero-Moser equation of motion obtained from the Lax formalism, equation (13) or the above, with the Hamiltonian equation of motion. The general proof of this consistency follow the similar lengthy steps as in ref.,[3] and thus it will not be reproduced here. Instead, two explicit cases will be examined.

In the matrix element evaluation of the Lax equation, it should be noted that the L operator is a Hermitean, whereas the M operator is an anti-Hermitean matrix. The diagonal part of the Lax equation (13) is calculated to be,

$$(L_t)_{\beta\beta} = \sum_{\kappa\in\Delta}[L_{\beta\kappa}M_{\kappa\beta} - M_{\beta\kappa}L_{\kappa\beta}], \qquad (26)$$

or explicitly,

$$\dot{p}\cdot\beta = 2\sum_{\kappa\in\Delta}\{[X_{\beta\kappa} + (X_r)_{\beta\kappa}][Y_{\kappa\beta} + (Y_r)_{\kappa\beta}]\}. \qquad (27)$$

The two explicit examples are the simplest doubly-laced Lie algebras, i.e. B_2 and C_3 cases.

4.1. B_2 case

The root space of this algebra is divided into two sets,

$$\begin{aligned}\Delta_s &= \{\pm\beta_1, \pm(\beta_1+\beta_2)\},\\ \Delta_l &= \{\pm\beta_2, \pm(2\,\beta_1+\beta_2)\}.\end{aligned}$$

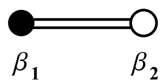

Figure 1. Dynkin diagram of the B_2 or $SO(5)$ Lie algebra.

For $\beta = (\beta_1+\beta_2) \in \Delta_s$, the equation of motion obtained from the Hamiltonian formalism, equation (8), yield

$$\begin{aligned}&\dot{p}\cdot(\beta_1+\beta_2)\\ &= -2\,\{\,2\,g^2\,x((2\,\beta_1+\beta_2)\cdot q)\,y((2\,\beta_1+\beta_2)\cdot q)\\ &\quad + 2\,g^2\,x((\beta_1+\beta_2)\cdot q)\,y((\beta_1+\beta_2)\cdot q)\\ &\quad + 2\,g^2\,x(\beta_2\cdot q)\,y(\beta_2\cdot q)\}, \qquad (28)\end{aligned}$$

and the Lax formalism, equation (27), yield

$$\begin{aligned}&\dot{p}\cdot(\beta_1+\beta_2)\\ &= -2\,\{\,g'^2\,x((2\,\beta_1+\beta_2)\cdot q)\,y((2\,\beta_1+\beta_2)\cdot q)\\ &\quad + 2\,g^2\,x((\beta_1+\beta_2)\cdot q)\,y((\beta_1+\beta_2)\cdot q)\\ &\quad + g'^2\,x(\beta_2\cdot q)\,y(\beta_2\cdot q)\}. \qquad (29)\end{aligned}$$

In the above the non-trivial solution of the E_d and $E_{d'}$ operators in equation (27) are obtained from the following table

β	γ	α
β_2	$\pm(2\,\beta_1+\beta_2)$	$-\beta_1$
		$(\beta_1+\beta_2)$
$(2\,\beta_1+\beta_2)$	$\pm\,\beta_2$	β_1
		$(\beta_1+\beta_2)$

It is readily seen that the two equation of motions are identical provided the coupling parameters are related as $g' = g\sqrt{2}$.

Further, for $\beta = (2\,\beta_1+\beta_2) \in \Delta_l$, the equation of motion obtained from the Hamiltonian formalism, equation (8), yield

$$\begin{aligned}&\dot{p}\cdot(2\beta_1+\beta_2)\\ &= -2\,\{\,2\,g^2\,x(\beta_1\cdot q)\,y(\beta_1\cdot q)\\ &\quad + 4\,g^2\,x((2\,\beta_1+\beta_2)\cdot q)\,y((2\,\beta_1+\beta_2)\cdot q)\\ &\quad + 2\,g^2\,x((\beta_1+\beta_2)\cdot q)\,y((\beta_1+\beta_2)\cdot q)\}, \qquad (30)\end{aligned}$$

and the Lax formalism, equation (27), yield

$$\begin{aligned}&\dot{p}\cdot(2\beta_1+\beta_2)\\ &= -2\,\{\,2\,g'^2\,x((2\,\beta_1+\beta_2)\cdot q)\,y((2\,\beta_1+\beta_2)\cdot q)\\ &\quad + 2\,g^2\,x((\beta_1+\beta_2)\cdot q)\,y((\beta_1+\beta_2)\cdot q)\\ &\quad + 2\,g^2\,x(\beta_1\cdot q)\,y(\beta_1\cdot q)\}. \qquad (31)\end{aligned}$$

As in the previous case,the non-trivial solution of the E_d and $E_{d'}$ operators in equation (27) are obtained from the table above, and again the two equation of motions are consistent when g' is chosen to be equal to $g\sqrt{2}$.

4.2. C_3 case

The root space of this algebra is divided into two sets,

$$\begin{aligned}\Delta_s &= \{\pm\beta_2, \pm\beta_3, \pm(\beta_1+\beta_2), \pm(\beta_2+\beta_3),\\ &\quad \pm(\beta_1+\beta_2+\beta_3), \pm(\beta_1+2\,\beta_2+\beta_3)\},\\ \Delta_l &= \{\pm\beta_1, \pm(\beta_1+\,2\,\beta_2), \pm(\beta_1+\,2\,\beta_2+\,2\,\beta_3)\}.\end{aligned}$$

For $\beta = \beta_2 \in \Delta_s$, the equation of motion obtained

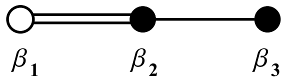

Figure 2. Dynkin diagram of the C_3 or $Sp(3)$ Lie algebra.

from the Hamiltonian formalism, equation (8), yield

$$\begin{aligned}
&\dot{p}\cdot\beta_2 \\
&= -2\,\{-2\,g^2\,x(\beta_1\cdot q)\,y(\beta_1\cdot q) \\
&\quad +2\,g^2\,x(\beta_2\cdot q)\,y(\beta_2\cdot q) - g^2\,x(\beta_3\cdot q)\,y(\beta_3\cdot q) \\
&\quad + g^2\,x((\beta_2+\beta_3)\cdot q)\,y((\beta_2+\beta_3)\cdot q) \\
&\quad +2\,g^2\,x((\beta_1+\,2\,\beta_2)\cdot q)\,y((\beta_1+\,2\,\beta_2)\cdot q) \\
&\quad - g^2\,x((\beta_1+\beta_2+\beta_3)\cdot q)\,y((\beta_1+\beta_2+\beta_3)\cdot q) \\
&\quad + g^2\,x((\beta_1+\,2\,\beta_2+\beta_3)\cdot q)\,y((\beta_1+\,2\,\beta_2+\beta_3)\cdot q)\},
\end{aligned} \tag{32}$$

and the Lax formalism, equation (27), yield

$$\begin{aligned}
&\dot{p}\cdot\beta_2 \\
&= -2\,\{\,2\,g^2\,x(\beta_2\cdot q)\,y(\beta_2\cdot q) \\
&\quad + g^2\,x((\beta_2+\beta_3)\cdot q)\,y((\beta_2+\beta_3)\cdot q) \\
&\quad + g'^2\,x((\beta_1+\,2\,\beta_2)\cdot q)\,y((\beta_1+\,2\,\beta_2)\cdot q) \\
&\quad + g^2\,x((\beta_1+\,2\,\beta_2+\beta_3)\cdot q)\,y((\beta_1+\,2\,\beta_2+\beta_3)\cdot q) \\
&\quad - g'^2\,x(\beta_1\cdot q)\,y(\beta_1\cdot q) - \,g^2\,x(\beta_3\cdot q)\,y(\beta_3\cdot q) \\
&\quad - g^2\,x((\beta_1+\beta_2+\beta_3)\cdot q)\,y((\beta_1+\beta_2+\beta_3)\cdot q)\}.
\end{aligned} \tag{33}$$

In the above the non-trivial solution of the E_d and $E_{d'}$ operators in equation (27) are obtained from the following table

β	γ	α
β_1	$\pm(\beta_1+\,2\,\beta_2)$	$-\beta_2$
		$(\beta_1+\beta_2)$
	$\pm(\beta_1+\,2\,\beta_2\,2\,\beta_3)$	$-(\beta_2+\beta_3)$
		$(\beta_1+\beta_2+\beta_3)$
$(\beta_1+\,2\,\beta_2)$	$\pm\,\beta_1$	β_2
		$(\beta_1+\beta_2)$
	$\pm(\beta_1+\beta_2+\beta_3)$	$-\beta_3$
		$(\beta_1+\,2\,\beta_2+\beta_3)$
$(\beta_1+\,2\,\beta_2+\,2\,\beta_3)$	$\pm\,\beta_1$	$(\beta_2+\beta_3)$
		$(\beta_1+\beta_2+\beta_3)$
	$\pm(\beta_1+\,2\,\beta_2)$	β_3
		$(\beta_1+\,2\,\beta_2+\beta_3)$

It is readily seen that the two equation of motions are identical provided the coupling parameters are related as $g' = g\sqrt{2}$.

Further, for $\beta = (\beta_1+2\,\beta_2)\in\Delta_l$, the equation of motion obtained from the Hamiltonian formalism, equation (8), yield

$$\begin{aligned}
&\dot{p}\cdot(\beta_1+\,2\,\beta_2) \\
&= -2\,\{\,2\,g^2\,x(\beta_2\cdot q)\,y(\beta_2\cdot q) - \,2\,g^2\,x(\beta_3\cdot q)\,y(\beta_3\cdot q) \\
&\quad +2\,g^2\,x((\beta_1+\beta_2)\cdot q)\,y((\beta_1+\beta_2)\cdot q)\,+ \\
&\quad +4\,g^2\,x((\beta_1+\,2\,\beta_2)\cdot q)\,y((\beta_1+\,2\,\beta_2)\cdot q) \\
&\quad +2\,g^2\,x((\beta_1+\,2\,\beta_2+\beta_3)\cdot q)\,y((\beta_1+\,2\,\beta_2+\beta_3)\cdot q)\},
\end{aligned} \tag{34}$$

and the Lax formalism, equation (27), yield

$$\begin{aligned}
&\dot{p}\cdot(\beta_1+\,2\,\beta_2) \\
&= -2\,\{\,2\,g^2\,x((\beta_1+\beta_2)\cdot q)\,y((\beta_1+\beta_2)\cdot q) \\
&\quad +2\,g^2\,x((\beta_1+\,2\,\beta_2+\beta_3)\cdot q)\,y((\beta_1+\,2\,\beta_2+\beta_3)\cdot q) \\
&\quad +2\,g'^2\,x((\beta_1+\,2\,\beta_2)\cdot q)\,y((\beta_1+\,2\,\beta_2)\cdot q) \\
&\quad +2\,g^2\,x(\beta_2\cdot q)\,y(\beta_2\cdot q) - \,2\,g^2\,x(\beta_3\cdot q)\,y(\beta_3\cdot q)\}.
\end{aligned} \tag{35}$$

As in the previous case,the non-trivial solution of the E_d and $E_{d'}$ operators in equation (27) are obtained from the table above, and again the two equation of motions are consistent when g' is chosen to be equal to $g\sqrt{2}$.

5. Conclusion

From the above exposition, one can conclude that the Lax pair operator introduced in this work, equation (14)-(22), is associated with the Calogero-Moser model based on the non simply-laced Lie algebras, to be precise the doubly-laced cases. Further, one can also conclude that from the existence of the Lax pair operators, the Calogero-Moser models based on the doubly-laced Lie algebras are integrable. The proposed Lax pair operator returns back to the Lax pair operator proposed by Bordner et.al.,[3] when applied to the simply-laced Lie algebra cases. A natural generalization to include the triply-laced case of the G_2 Lie algebra is straightforward.

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