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Blow-up Dynamics of Higher Dimensional Klein-Gordon Equation with Nonminimal Coupling in Subcritical Case

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Abstract

The aim of this present work is to study the blow-up dynamics and lifespan estimates for solution to higher dimensional Klein-Gordon Equation in subcritical case, in which $1 < p < p_{sc}$. We construct the equation of motion from the Lagrangian of Klein-Gordon with non-minimal coupling, where the coupling interaction of the scalar field is proportional to the scalar curvature of the spacetime. The equation of motion has the form like nonlinear damped wave equation with mass. The novelty of this work is the time dependent of nonlinear term. We use test function method to proof the lifespan estimate.

Keywords: Blow up dynamics, Klein-Gordon equation, Nonminimal coupling

INTRODUCTION

Consider the following nonlinear damped wave equation with mass

$$u_{tt} - \Delta u + \mu_1 (1+t)^{\alpha} u_t + \mu_2^2 (1+t)^{\beta} u = \mathcal{F}(u) \quad (1)$$

with $\mu_1, \mu_2^2 \in R$. we classify the damping term as: overdamping case ($\alpha > 1$), scattering case ($\alpha < -1$), effective case ($-1 < \alpha \leq 1$), and scale-invariant case ($\alpha = -1$). In recent years, the study of Eq.(1) has become an interesting and unfinished problem including the wellposedness and blow-up dynamics with various assumptions of damping, mass, and nonlinear term, i.e effective case [2] , scattering case [3,5], scale-invariant case [6,7] and also the references therein.

In this paper, we study the blow-up dynamics of higher dimensional Klein-Gordon equation with nonminimal coupling which has the similar form with Eq.(1). We focused on the scale-invariant case in which $\alpha = -1$ (and $\beta = -2$). Let us consider the higher dimensional spatially flat spacetime which can be constructed by d-dimensional spatially flat Lorentzian manifold denoted by $\mathcal{M}^d$, $d \geq 4$ with standard coordinates $x^{\mu} = (x^0 = t,\ x^i)$, $\mu = 0,1,\dots,d-1$, and $i = 1,2,\dots,d-1$ equipped with the Lorentzian metric with the signature $\{-1,1,\dots,1\}$. We write down the metric as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t) \sum_{i=1}^{d-1} \mathrm{d}x_i^2, \quad (2)$$

with x^i denotes the Cartesian coordinatees for R^{d-1}.

Defining a new time coordinate denoted by τ which follows the conformal transformation

$$\frac{\mathrm{d}\tau}{dt} = \frac{1}{a(t)}, \quad (3)$$

thus we rewrite the metric (2) as

$$\mathrm{d}s^2 = a^2(\tau)\left(-\mathrm{d}\tau^2 + \sum_{i=1}^{d-1} \mathrm{d}x_i^2\right). \quad (4)$$

Now our spactime $\mathcal{M}^d$ is conformal to flat Minkowski space $\mathcal{M}^{d+1} \approx R \times R^{d-1}$. From the

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metric (2) we get Ricci tensor and the scalar curvature

$$R_{\mu\nu} = \dot{H}\left(\eta_{\mu\nu} - (d-2)\delta_\mu^0\delta_\nu^0\right) + H^2(d-2)\left(\eta_{\mu\nu} + \delta_\mu^0\delta_\nu^0\right), \qquad (5)$$

$$R = (d-1)a^{-2}\left(2\dot{H} + (d-2)H^2\right), \qquad (6)$$

assuming that the scale factor $a = a(\tau)$ belongs to $\mathcal{C}^n$ function with $n \geq 2$, for all $\tau > 0$. Moreover, we define the Hubble parameter $H \equiv \frac{\dot{a}}{a}$ with $\dot{a} \equiv \frac{\mathrm{d}a}{\mathrm{d}\tau}$.

Next, we define the action of real scalar field on the higher dimensional spatially flat universe as a background with additional non-minimal coupling where the coupling interaction of the scalar field φ is proportional to the scalar curvature of the spacetime. The action of real scalar field has the form

$$S = \int \mathrm{d}\tau \mathrm{d}x\sqrt{-g}\left(\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{\xi}{2}R\varphi^2 - V(\varphi)\right), \qquad (7)$$

where g and R are the determinant and the scalar curvature of the metric (4). The second term of the R.H.S of (7) describes the non-minimal coupling with positive constant ξ. The real smooth function $V(\varphi)$ denotes the scalar potential which satisfy the condition $V(0) = 0$ dan $\partial_\varphi V(0) = 0$.

Throughout this paper, we use some assumptions as follows:

a. The damping term is describe as the Hubble parameter varies of time

$$H(\tau) = H_0(1+\tau)^{-1}. \qquad (8)$$

b. The scalar potential has the form

$$V(\varphi) = -\frac{\epsilon}{p+1}|\varphi|^{p+1}, \qquad (9)$$

with ϵ is a small positive parameter and $p > 1$.

Thus we have,

$$\begin{aligned}&\varphi_{\tau\tau} - \Delta\varphi + \frac{(d-2)H_0}{1+\tau}\varphi_\tau \\ &+ \frac{\xi(d-1)(-2H_0 + (d-2)H_0^2)}{(1+\tau)^2}\varphi \\ &= \epsilon\,(1+\tau)^{2H_0}\,|\varphi|^p, \\ &\varphi(\tau_0, x) = f(x) \in \left(R^{d-1}\right), \\ &\varphi_\tau(\tau_0, x) = g(x) \in \left(R^{d-1}\right), \\ &\text{for all } \tau > \tau_0,\ x \in R^{d-1},\end{aligned} \qquad (10)$$

Let us introduce a new parameter

$$\delta = [(d-2)H_0 - 1]^2 - 4\xi(d-1)[-2H_0 + (d-2)H_0^2] \qquad (11)$$

describing the relation of damping and mass term. In the case $\delta \geq 0$, the damping term is more dominant than the mass term.

In the previous works [1,4,8], the wellposedness problem of (10) has been studied. In the present work, we study the blow-up dynamics and lifespan estimate for solution in subcritical case, in which $1 < p < p_{sc}$. The novelty of this work is the time dependent of nonlinear term.

MAIN RESULTS

We introduce a notion of energy solution by the following definition.

Definition 1. *Let* $f \in H^1\left(R^{d-1}\right)$ *and* $g \in L^2\left(R^{d-1}\right)$. *We define* φ *as the solution to* (10) *on* $[\tau_0, T)$ *if*

$$\varphi \in \mathcal{C}\left([\tau_0, T), H^1(R^{d-1})\right) \cap \mathcal{C}^1\left([\tau_0, T), L^2(R^{d-1})\right), \qquad (12)$$

satisfies

$$\begin{aligned}&\int_{R^{d-1}} \varphi_\tau(\tau, x)\phi(\tau, x)\mathrm{d}x - \int_{R^{d-1}} \varphi_\tau(\tau_0, x)\phi(\tau_0, x)\mathrm{d}x \\ &- \int_{\tau_0}^{\tau}\int_{R^{d-1}} \varphi_\tau(s, x)\phi_\tau(s, x)\,\mathrm{d}x\mathrm{d}s \\ &+ \int_{\tau_0}^{\tau}\int_{R^{d-1}} \nabla\varphi(s, x)\,.\,\nabla\phi(s, x)\,\mathrm{d}x\mathrm{d}s \\ &+ \int_{\tau_0}^{\tau}\int_{R^{d-1}}\left(\frac{(d-2)H_0}{1+s}\varphi_\tau(s, x) + \frac{\xi(d-1)\left(-2H_0 + (d-2)H_0^2\right)}{(1+s)^2}\varphi(s, x)\right)\phi(s, x)\,\mathrm{d}x\mathrm{d}s \\ &= \epsilon\int_{\tau_0}^{\tau}\int_{R^{d-1}} (1+s)^{2H_0}|\varphi(s, x)|^p\,\phi(s, x)\,\mathrm{d}x\mathrm{d}s,\end{aligned} \qquad (13)$$

for any $\phi \in \mathcal{C}_0^\infty\left([\tau_0, T) \times R^{d-1}\right)$ *and* $\tau \in [\tau_0, T)$.

We state the blow-up dynamics and lifespan estimates for the solution of (10) in subcritical case as follows.

Theorem 1. *Let* $d \geq 4$. *We assume that the initial data* $f \in H^1\left(R^{d-1}\right)$ *and* $g \in L^2\left(R^{d-1}\right)$ *are compacly supported in* $B_R \coloneqq \{x \in R^{d-1}: |x| \leq R\}$, *and non identically zero. We define* $\delta \geq 0$ *as in* (11) *with non negative damping coefficient and mass such that* $H_0 \geq \frac{2}{d-2}$.

Let φ *be a solution of* (10). *In subcritical case* $1 < p < p_{sc}$, *we obtain that* φ *blow up in finite time with lifespan* $T = T(\epsilon)$. *Furthermore,* $T(\epsilon)$ *fullfiles the estimate*

$$T(\epsilon) \leq C\,\epsilon^{-\frac{2(p-1)}{\gamma_{sc}}}, \qquad (14)$$

with

$$\gamma_{sc} \coloneqq -dp^2 + \left(\frac{8}{d-2} + d\right)p + 4, \qquad (15)$$

where C is a positive constant independent of ϵ. In addition, p_{sc} is the positive root of $\gamma_{sc}=0$ as

$$p_{sc}=\frac{\frac{8}{d-2}+d+\sqrt{\left(\frac{8}{d-2}+d\right)^2+16d}}{2d}. \qquad (16)$$

TEST FUNCTION FOR SUBCRITICAL CASE

We will proof Theorem 1 using test function method. First, we define the modified Bessel function of the second kind of order ν

$$\mathcal{K}_\nu(\tau)=\int_{\tau_0}^{\infty}\exp(-\tau\cosh z)\cosh(\nu z)\,\mathrm{d}z, \qquad (17)$$

$\nu\in R^{d-1}$, fullfiles the equation

$$\left(\tau^2\frac{d^2}{d\tau^2}+\tau\frac{d}{d\tau}-(\tau^2-\nu^2)\right)\mathcal{K}_\nu(\tau)\ =\ 0, \qquad (18)$$

for $\tau>0$.

We define the auxiliary function with respect to the time variable

$$\lambda(\tau):=(1+\tau)^{\frac{(d-2)H_0+1}{2}}\mathcal{K}_{\frac{\sqrt{\delta}}{2}}(1+\tau),\tau\geq 0, \qquad (19)$$

fullfiles the equation

$$(1+\tau)^2\frac{d^2\lambda(\tau)}{d\tau^2}-(d-2)H_0(1+\tau)\frac{d\lambda(\tau)}{d\tau}+\Big((d-2)H_0+\xi(d-1)(-2H_0+(d-2)H_0^2)-(1+\tau)^2\Big)\lambda(\tau)=0,\tau>0. \qquad (20)$$

Then, we define

$$\psi(x):=\int_{S^{d-2}}e^{x.\omega}\,\mathrm{d}\omega, \qquad (21)$$

which satisfies

$$\Delta\psi(x)=\psi(x), \qquad (22)$$

and the asymptotic estimate

$$\psi(x)\sim C_{d-1}|x|^{-\frac{(d-2)}{2}}e^{|x|} \qquad (23)$$

as $|x|\to\infty$. Now, we define the test function for the subcritical case

$$\Psi(\tau,x):=\lambda(\tau)\psi(x). \qquad (24)$$

We use the test function to derive a lower bound for $|\varphi|^p$ in the following Lemma.

Lemma 1. *We define the initial data f and g such that $supp(f,g)\subset B_R$ with $R>0$. Then, we have a local energy solution φ satisfies*

$$supp\ \varphi\subset\{(\tau,x)\in[\tau_0,T)\times R^{d-1}:|x|\leq\tau+R\}. \qquad (25)$$

There exists τ_0 independent of f and g, such that for any $\tau>\tau_0$ and $p>1$, it satisfies the estimate

$$C(1+\tau)^{\frac{d-2}{2}[2-(H_0+1)p]}\leq\int_{R^{d-1}}|\varphi|^p\,dx, \qquad (26)$$

where $C=C(f,g,\psi,p,R)>0$.

Proof. We claim that the support of $\varphi(.,\tau)$ is contained in $B(\tau_0,\tau+R)$ since the supports of f and g are contained in $B(\tau_0,R)$. Hence, the statement (25) is fullfiled. Then, we define functional

$$F(\tau):=\int_{R^{d-1}}\varphi(\tau,x)\Psi(\tau,x)\mathrm{d}x. \qquad (27)$$

By Hölder inequality we obtain

$$|F(\tau)|^p\left(\int_{|x|\leq\tau+R}\Psi^{p'}(\tau,x)\mathrm{d}x\right)^{-(p-1)}\leq\int_{R^{d-1}}|\varphi(\tau,x)|^p\,\mathrm{d}x, \qquad (28)$$

with $\frac{1}{p}+\frac{1}{p'}=1$.

Next, we determine a lower bound for $|F(\tau)|^p$ and an upper bound for

$$\left(\int_{|x|\leq\tau+R}\Psi^{p'}(\tau,x)\mathrm{d}x\right)^{-(p-1)}.$$

From the definition of energy solution, we have

$$\int_{\tau_0}^{\tau}\int_{R^{d-1}}\varphi_{ss}\,\Psi\mathrm{d}x\mathrm{d}s-\int_{\tau_0}^{\tau}\int_{R^{d-1}}\varphi\Delta\Psi\,\mathrm{d}x\mathrm{d}s+\int_{\tau_0}^{\tau}\int_{R^{d-1}}\partial_s\left(\frac{(d-2)H_0}{1+s}\Psi\varphi\right)-\partial_s\left(\frac{(d-2)H_0}{1+s}\Psi\right)\varphi+\frac{\xi(d-1)(-2H_0+(d-2)H_0^2)}{(1+s)^2}\Psi\varphi\,\mathrm{d}x\mathrm{d}s$$
$$=\epsilon\int_{\tau_0}^{\tau}\int_{R^{d-1}}(1+s)^{2H_0}|\varphi(s,x)|^p\,\Psi\,\mathrm{d}x\mathrm{d}s. \qquad (29)$$

Using the relation $\Delta\psi=\psi$ and definition (20), we obtain

$$\int_{R^{d-1}}\left(\varphi_s\Psi-\Psi_s\varphi+\frac{(d-2)H_0}{1+s}\varphi\Psi\right)\mathrm{d}x\Big|_{\tau_0}^{\tau}=\epsilon\int_{\tau_0}^{\tau}\int_{R^{d-1}}(1+s)^{2H_0}|\varphi(s,x)|^p\,\Psi\,\mathrm{d}x\mathrm{d}s. \qquad (30)$$

Since the R.H.S of (30) is positive, we obtain

$$F'(\tau)+\left(\frac{(d-2)H_0}{1+\tau}-2\frac{\lambda'(\tau)}{\lambda(\tau)}\right)F(\tau)\geq C_{f,g}, \qquad (31)$$

with

$$C_{f,g}=\int_{R^{d-1}}\left(g(x)\lambda(\tau_0)+\left(\frac{(d-2)H_0}{1+\tau_0}\lambda(\tau_0)-\lambda'(\tau_0)\right)f(x)\right)\psi(x)\mathrm{d}x. \qquad (32)$$

Multiplying both sides with $\frac{(1+\tau)^{(d-2)H_0}}{\lambda^2(\tau)}$, and integrating over $[\tau_0,\tau]$ such that

$$F(\tau)\geq\left(\frac{\lambda(\tau)}{\lambda(\tau_0)}\right)^2\left(\frac{1+\tau_0}{1+\tau}\right)^{(d-2)H_0}F(\tau_0)+\int_{\tau_0}^{\tau}\int_{R^{d-1}}\partial_s\left(\frac{(d-2)H_0}{1+s}\Psi\varphi\right)-\partial_s\left(\frac{(d-2)H_0}{1+s}\Psi\right)\varphi+C_{f,g}\frac{\lambda^2(\tau)}{(1+\tau)^{(d-2)H_0}}\int_{\tau_0}^{\tau}\frac{(1+s)^{(d-2)H_0}}{\lambda^2(s)}\mathrm{d}s. \qquad (33)$$

Moreover, using the derivative identity of modified Bessel function we obtain

$$\frac{(d-2)\mathrm{H}_0}{1+\tau_0}\lambda(\tau_0)-\lambda'(\tau_0)=\frac{(d-2)H_0-1-\sqrt{\delta}}{2}$$
$$(1+\tau_0)^{\frac{(d-2)\mathrm{H}_0}{2}-1}\mathcal{K}_{\frac{\sqrt{\delta}}{2}}(1+\tau_0)$$
$$+(1+\tau_0)^{\frac{(d-2)H_0+1}{2}}\mathcal{K}_{\frac{\sqrt{\delta}}{2}+1}(1+\tau_0). \tag{34}$$

Since f and g are compactly supported initial data, hence $C_{f,g}$ is finite and positive. Using the definition of (19) and $F(\tau_0)\geq 0$, we obtain the lower bound for $|F(\tau)|^p$ as follows

$$|F(\tau)|^p \geq C_{f,g}^p(1+\tau)^p\mathcal{K}_{\frac{\sqrt{\delta}}{2}}^{2p}(1+\tau)$$
$$\times\left(\int_{\tau_0}^{\tau}\frac{\mathrm{d}s}{(1+s)\mathcal{K}_{\frac{\sqrt{\delta}}{2}}^2(1+s)}\right)^p\geq 0. \tag{35}$$

Then, we estimate

$$\left(\int_{|x|\leq\tau+R}\Psi^{p'}(\tau,x)\mathrm{d}x\right)^{-(p-1)}$$
$$\leq C_{\psi,R}^{-(p-1)}(1+\tau)^{-(d-2)(p-1)-\frac{(d-2)(H_0-1)+1}{2}p}$$
$$\times e^{-p(\tau+R)}\mathcal{K}_{\frac{\sqrt{\delta}}{2}}^{-p}(1+\tau), \tag{36}$$

where $C_{\psi,R}$ is a positive constant.

Combining the estimate (28), (35), and (36), we get

$$C_{f,g}^pC_{\psi,R}^{1-p}(1+\tau)^{p-(d-2)(p-1)-\frac{(d-2)(H_0-1)+1}{2}p}$$
$$e^{-p(\tau+R)}\mathcal{K}_{\frac{\sqrt{\delta}}{2}}^{p}(1+\tau)\left(\int_{\tau_0}^{\tau}\frac{\mathrm{d}s}{(1+s)\mathcal{K}_{\frac{\sqrt{\delta}}{2}}^2(1+s)}\right)^p$$
$$\leq\int_{R^{d-1}}|\varphi|^p\mathrm{d}x. \tag{37}$$

Now, we use the limiting behavior of $\mathcal{K}_\nu(\tau)$ such that

$$\mathcal{K}_{\frac{\sqrt{\delta}}{2}}^{p}(1+\tau)\sim\left(\frac{\pi}{2(1+\tau)}\right)^{\frac{p}{2}}e^{-p(\tau+1)}, \tag{38}$$

and

$$\int_{\tau_0}^{\tau}\frac{1}{(1+s)\mathcal{K}_{\frac{\sqrt{\delta}}{2}}^2(1+s)}\mathrm{d}s\geq\frac{1}{2\pi}e^{2(1+\tau)}. \tag{39}$$

Finally, we rewrite the estimate (37) as follows

$$C_1(1+\tau)^{\frac{d-2}{2}[2-(H_0+1)p]}\leq\int_{R^{d-1}}|\varphi|^p\mathrm{d}x,$$

where $C_1 := 2^{-\frac{3p}{2}}C_{f,g}^pC_{\psi,R}^{1-p}e^{p(1-R)}\pi^{-\frac{p}{2}}$.

This completes the proof of Lemma 1. ■

PROOF OF THEOREM 1

We start the proof by define the functional

$$G(\tau):=\int_{R^{d-1}}\varphi(\tau,x)\mathrm{d}x. \tag{40}$$

Then, we choose $\phi=\phi(s,x)$ satisfies $\phi\equiv 1$ in $\{(x,s)\in[\tau_0,\tau]\times R^{d-1}:|x|\leq s+R\}$, such that from (13) we have

$$G''(\tau)+\frac{(d-2)H_0}{1+\tau}G'(\tau)+\frac{\xi(d-1)\left(-2H_0+(d-2)H_0^2\right)}{(1+\tau)^2}G(\tau)$$
$$=\epsilon\int_{R^{d-1}}(1+s)^{2H_0}|\varphi(s,x)|^p\,\mathrm{d}x. \tag{41}$$

Let us define the quadratic equation

$$r^2-\left((d-2)H_0-1\right)r+$$
$$\xi(d-1)(-2H_0+(d-2)H_0^2)=0, \tag{42}$$

which has the roots

$$r_1=\frac{(d-2)H_0-1-\sqrt{\delta}}{2},\quad r_2=\frac{(d-2)H_0-1+\sqrt{\delta}}{2}, \tag{43}$$

with $\delta\geq 0$. Thus, we can rewrite (41) as follows

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(G'(\tau)+\frac{r_1}{1+\tau}G(\tau)\right)+\frac{r_2+1}{1+\tau}\left(G'(\tau)+\frac{r_1}{1+\tau}G(\tau)\right)$$
$$=\epsilon\int_{R^{d-1}}(1+s)^{2H_0}|\varphi(s,x)|^p\,\mathrm{d}x. \tag{44}$$

Multiplying both sides by $(1+\tau)^{r_2+1}$ and integrating over $[\tau_0,\tau]$, we obtain

$$(1+\tau)^{r_2+1}\left(G'(\tau)+\frac{r_1}{1+\tau}G(\tau)\right)-(1+\tau_0)^{r_2+1}$$
$$\times\left(G'(\tau_0)+\frac{r_1}{1+\tau_0}G(\tau_0)\right)$$
$$=\epsilon\int_{\tau_0}^{\tau}\int_{R^{d-1}}(1+s)^{2H_0+r_2+1}|\varphi|^p\,\mathrm{d}x\mathrm{d}s. \tag{45}$$

Since the initial data is non negative, we have

$$G'(\tau)+\frac{r_1}{1+\tau}G(\tau)>\epsilon(1+\tau)^{-(r_2+1)}$$
$$\times\int_{\tau_0}^{\tau}\int_{R^{d-1}}(1+s)^{2H_0+r_2+1}|\varphi|^p\,\mathrm{d}x\mathrm{d}s. \tag{46}$$

Multiplying both sides of above inequality by $(1+\tau)^{r_1}$ and integrating over $[\tau_0,\tau]$, we obtain

$$(1+\tau)^{r_1}G(\tau)-(1+\tau_0)^{r_1}G(\tau_0)>$$
$$\epsilon\int_{\tau_0}^{\tau}(1+t)^{r_1-(r_2+1)}\int_{\tau_0}^{t}\int_{R^{d-1}}(1+s)^{2H_0+r_2+1}|\varphi|^p\,\mathrm{d}x\mathrm{d}s\mathrm{d}t. \tag{47}$$

Furthermore, using the non negative condition of the initial data we obtain

$$G(\tau)\geq\epsilon(1+\tau)^{-r_1}\int_{\tau_0}^{\tau}(1+t)^{r_1-(r_2+1)}$$
$$\times\int_{\tau_0}^{t}\int_{R^{d-1}}(1+s)^{2H_0+r_2+1}|\varphi|^p\,\mathrm{d}x\mathrm{d}s\mathrm{d}t. \tag{48}$$

Substituting (26) to (48), we obtain

$$G(\tau)\geq\epsilon C_1(1+\tau)^{-r_1}\int_{\tau_0}^{\tau}(1+t)^{r_1-r_2-1}$$
$$\times\int_{\tau_0}^{t}(1+s)^{2H_0+r_2+1+\frac{d-2}{2}[2-(H_0+1)p]}\,\mathrm{d}s\,\mathrm{d}t$$
$$\geq\epsilon C_2(1+\tau)^{-r_2-1+2H_0-\frac{(d-2)}{2}(H_0+1)p}$$
$$\times(\tau-\tau_0)^{d+r_2+1}, \tag{49}$$

with $C_2 = \frac{C_1}{(d+r_2)(d+r_2+1)}$.

Now, we obtain lifespan estimates for solution (10) using iteration argument as follows. Applying Hölder inequality to (48), such that

$$G(\tau) \ge \epsilon C_0 (1+\tau)^{-r_1} \int_{\tau_0}^{\tau} (1+t)^{r_1-(r_2+1)} \times \int_{\tau_0}^{\tau} (1+s)^{2H_0+r_2+1+(d-1)(1-p)} |G(s)|^p \, \mathrm{d}s\mathrm{d}t, \quad (50)$$

where

$$C_0 := \left(\mathrm{meas}(B_1)\right)^{1-p} R^{-(d-1)(p-1)} > 0. \quad (51)$$

We assume that

$$G(\tau) \ge D_j (1+\tau)^{-a_j} (\tau-\tau_0)^{b_j}, \quad (52)$$

for $\tau > \tau_0$, $j = 1,2,3,\ldots$, and D_j, a_j, b_j to be determined later. For $j = 1$, from (49) we obtain

$$D_1 = \epsilon C_2, \quad (53)$$

$$a_1 = r_2 + 1 - 2H_0 + \frac{(d-2)}{2}(H_0+1)p, \quad (54)$$

$$b_1 = d + r_2 + 1. \quad (55)$$

Substituting (52) to (50), we obtain

$$G(\tau) \ge \epsilon C_0 (1+\tau)^{-r_1} \int_{\tau_0}^{\tau} (1+t)^{r_1-(r_2+1)} \times \int_{\tau_0}^{\tau} (1+s)^{2H_0+r_2+1+(d-1)(1-p)} \times D_j^p (1+s)^{-pa_j} (s-\tau_0)^{pb_j} \, \mathrm{d}s\mathrm{d}t. \quad (56)$$

Moreover, we use the same method in deriving (49) to obtain

$$G(\tau) \ge \frac{\epsilon C_0 D_j^p}{(r_2+pb_j+2)(r_2+pb_j+3)} \times (1+\tau)^{-r_2-1+2H_0+(d-1)(1-p)-pa_j} \times (\tau-\tau_0)^{r_2+pb_j+3}. \quad (57)$$

From (57) we have

$$D_{j+1} = \frac{\epsilon C_0}{(r_2+pb_j+2)(r_2+pb_j+3)} D_j^p, \quad (58)$$

$$a_{j+1} = r_2 + 1 - 2H_0 - (d-1)(1-p) + pa_j, \quad (59)$$

$$b_{j+1} = r_2 + pb_j + 3. \quad (60)$$

Hence, from (54), (55), (59), and (60), we obtain

$$a_j = \alpha p^{j-1} - \left[\frac{1}{p-1}(r_2+1-2H_0) + d - 1\right], \quad (61)$$

$$b_j = \beta p^{j-1} - \frac{r_2+3}{p-1}, \quad (62)$$

for any $j = 1,2,3,\ldots$, and the positive constants

$$\alpha = (r_2+1-2H_0)\left(\frac{p}{p-1}\right) + \frac{d-2}{2}(H_0+1)p + d - 1, \quad (63)$$

$$\beta = d + r_2 + 1 + \frac{r_2 + 3}{p-1}. \quad (64)$$

In addition, from (60) and (62) we have

$$b_{j+1} = r_2 + 3 + pb_j < p^j \beta. \quad (65)$$

Thus, from (58) we have

$$D_{j+1} \ge C_3 \frac{D_j^p}{p^{2j}}, \quad (66)$$

where

$$C_3 = \frac{\epsilon C_0}{\beta^2} = \frac{\epsilon C_0}{\left(d + r_2 + 1 + \frac{r_2+3}{p-1}\right)^2}. \quad (67)$$

Now, we calculate

$$\log D_j \ge p^{j-1} \log D_1 - 2\log p \sum_{k=1}^{j-1} kp^{j-1-k} + \log C_3 \sum_{k=1}^{j-1} p^k. \quad (68)$$

Using the relation

$$\sum_{k=1}^{j-1} kp^{j-1-k} = \frac{1}{p-1}\left(\frac{p^j-1}{p-1} - j\right), \quad (69)$$

and

$$\sum_{k=1}^{j-1} p^k = \frac{p-p^j}{1-p}, \quad (70)$$

we obtain

$$\log D_j \ge p^{j-1}\left(\log D_1 - \frac{2p\log p}{(p-1)^2} + \frac{p\log C_3}{p-1}\right) + \frac{2\log p}{p-1}\left(j + \frac{1}{p-1} - \frac{p\log C_3}{2\log p}\right). \quad (71)$$

For $j + \frac{1}{p-1} - \frac{p\log C_3}{2\log p} > 1$, we have

$$D_j \ge \exp\{p^{j-1}(\log D_1 - C_p)\}, \quad (72)$$

with $C_p := \frac{2p\log p}{(p-1)^2} - \frac{p\log C_3}{p-1}$.

Substituting (61), (62), and (72) to (52) we obtain

$$G(\tau) \ge \exp\left(p^{j-1}\mathcal{J}(\tau)\right)(1+\tau)^{\left[\frac{1}{p-1}(r_2+1-2H_0)+d-1\right]} \times (\tau-\tau_0)^{-\frac{r_2+3}{p-1}}, \quad (73)$$

where

$$\mathcal{J}(\tau) := \log D_1 - C_p - \alpha\log(1+\tau) + \beta\log(\tau-\tau_0). \quad (74)$$

For $\tau > 2\tau_0 + 1$, we obtain

$$\mathcal{J}(\tau) \ge \log\left(D_1(\tau-\tau_0)^{\beta-\alpha}\right) - C_p - \alpha\log 2, \quad (75)$$

where

$$\beta - \alpha = \frac{4 + 4H_0 p - (d-2)(H_0+1)p(p-1)}{2(p-1)}. \quad (76)$$

For $\mathcal{J}(\tau) > 1$, we obtain

$$\tau - \tau_0 > \left(\frac{e^{C_p+\alpha\log 2+1}}{D_1}\right)^{\alpha-\beta}. \quad (77)$$

From the estimate (73) we know that if $j \to \infty$ then $G(\tau) \to \infty$, and as the consequences $\varphi \to \infty$ in the finite time. In addition, we define $\delta \geq 0$ as in (11) with non negative damping coefficient and mass such that $H_0 \geq \frac{2}{d-2}$. Thus, we obtain

$$\tau > \max\left\{\tau_0 + \left(\frac{e^{C_p+\alpha \log 2+1}}{D_1}\right)^{\frac{2(p-1)}{\gamma_{sc}}}, 2\tau_0 + 1\right\}, \tag{78}$$

with

$$\gamma_{sc} := -dp^2 + \left(\frac{8}{d-2} + d\right)p + 4. \tag{79}$$

Finally, we obtain lifespan $T = T(\epsilon)$ of φ fullfiles

$$T(\epsilon) \leq C_4 \epsilon^{-\frac{2(p-1)}{\gamma_{sc}}}, \tag{80}$$

with $C_4 := \left(\frac{e^{C_p+\alpha \log 2+1}}{C_2}\right)^{\frac{2(p-1)}{\gamma_{sc}}}$. We define p_{sc} as the positive root of $\gamma_{sc} = 0$ as follows,

$$p_{sc} = \frac{\frac{8}{d-2} + d + \sqrt{\left(\frac{8}{d-2} + d\right)^2 + 16d}}{2d}, \tag{81}$$

such that for $1 < p < p_{sc}$, the solution φ blow up in finite time. This completes the proof of Theorem 1.

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