Indonesian Journal of Physics
Vol. 32. No.2, 29, 12, 2021

Journal homepage: http://ijphysics.com

Scalar-Torsion Theories in Four Dimensional Static Spacetimes

Mulyanto[1*], Fiki T Akbar[1,2], Bobby E Gunara[1,2]

[1]*Theoretical High Energy Physics Research Division, Institut Teknologi Bandung*
[2]*Indonesia Center for Theoretical and Mathematical Physics (ICTMP)*

(Received: December 23, 2021, Revised: December 29, 2021, Accepted: December 29, 2021)

Abstract

In this paper, we consider a class of static spacetimes scalar-torsion theories in four dimensioanal static spacetimes with the scalar potential turned on. We discover that the 2-dimensional submanifold must admit constant triplet structures, one of which is the torsion scalar. This indicates that these equations of motion can be reduced to a single highly non-linear ordinary differential equation known as the master equation. Then, we show that there are no exact solution of the scalar-torsion theory in four dimensions considering the Sinh-Gordon potential.

Keywords: Scalar-Torsion, Scalar Field, Modified Gravity, Master Equation.

INTRODUCTION

Teleparallel gravity is a gravitational interaction theory in which gravitation is attributed to torsion rather than curvature, as in general relativity (GR). The field equation of motions can be thought of as a force equation, similar to the Lorentz force equation in Maxwell electrodynamics, implying that there are no geodesic equations in the theory, as shown in [1]. Due to several developments in the cosmological context, teleparallel theories have been actively studied over the last decade. For example, the general teleparallel formulation of gravity, $f(T)$, where T is the torsion scalar, has been proposed as an alternative gravitational theory that yields several intriguing cosmological models [2].

Keeping track of the GR limit is an interesting overall question in generalised theories of gravity. Considering this theory in four dimensions, some authors show that the static spherically symmetric black holes do exist, see for example [3, 4, 5]. In [6], they also demonstrate that gravitational waves have the same polarization modes as General Relativity if the boundary term is minimally coupled to the torsion scalar and the scalar field.

Our particular interest is in considering a minimal $f(T)$ gravity coupled to a scalar field, where $f(T) = T$. The authors of [7] developed a scalar-torsion theory in which the torsion is coupled to a scalar with non-minimal derivative coupling and the scalar potential is activated. They then discussed the static spherically symmetric solutions of four-dimensional scalar-torsion theories under the assumption that the potential satisfies the exponential equation and examined the asymptotic solution of the master equation. However, there are some critical errors that have been corrected in [8] to produce the correct master equation. This implies that there is no wormhole-like solution for a specific form of the scalar potential, as claimed in [7].

The objective of this paper is to obtain the exact solution of scalar-torsion theories in four dimensional static spacetimes where $f(T) = T$ is non-minimally coupled with the kinetic terms of a real scalar field ϕ. We consider some exact potensial such as sinh-Gordon potential.

SCALAR-TORSION THEORIES IN FOUR DIMENSIONS

To begin, we will define a quantity called torsion on a D-dimensional spacetime whose form is given by

* Corresponding author.

E-mail address: mulyanto37@gmail.com

$$T^{\lambda}{}_{\mu\nu} = \omega^{\lambda}{}_{\nu\mu} - \omega^{\lambda}{}_{\mu\nu} \qquad (1)$$

where $\omega^{\lambda}{}_{\nu\mu}$ is called alternative connection with Greek indices run over $\mu,\nu,\lambda = 0,1,...,3$. In the scalar torsion theories, the Ricci tensor and Torsion scalar has the form

$$\mathrm{R} = g^{\mu\nu}\mathrm{R}_{\mu\nu} = -T + 2\nabla_{\mu}T_{\nu}{}^{\nu\mu} \qquad (2)$$

$$T = \frac{1}{4}T^{\mu\nu\lambda}T_{\mu\nu\lambda} + \frac{1}{2}T^{\mu\nu\lambda}T_{\lambda\nu\mu} - T_{\nu}{}^{\nu\mu}T^{\lambda}{}_{\lambda}\mu \qquad (3)$$

Let us now shortly discuss the equations of motion in the scalar-torsion theory in which it contains the non-minimal derivative coupling term. The scalar-torsion theory's action with non-minimal derivative coupling takes the following form:

$$S = -\frac{1}{2\kappa_4^2}\int d^4x\, eT - \int d^4x e\left[\left(\frac{1}{2} - \xi T\right)g^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi + V\right] \qquad (4)$$

where $\xi > 0$ is a coupling parameter and *e* is defined as the vielbein determinant. The coupling $\kappa_4 \equiv \frac{1}{M_p}$, M_p is the 4-dimensional Planck mass with $\xi > M_p$.

Varying (4) with respect to the vielbein e_a and its scalar field ϕ, then we can get the equation of motion as follows

$$\begin{aligned}&\left(\frac{2}{\kappa_D^2} - 4\xi\phi_{,\rho}\phi^{,\rho}\right)\left[(eS_{\kappa}{}^{\lambda\nu}e_{\bar{b}}{}^{\kappa})_{,\nu}\, e^{\bar{b}}{}_{\mu} + e\left(\frac{1}{4}T\delta^{\lambda}_{\mu} - S^{\nu\kappa\lambda}T_{\nu\kappa\mu}\right)\right]\\ &+4\xi\left[\frac{1}{2}eT\phi_{,\mu}\phi^{,\lambda} + eS_{\mu}{}^{\nu\lambda}(\phi_{,\kappa}\phi^{,\kappa})_{,\nu}\right]\\ &+e\left(\frac{1}{2}\phi_{,\rho}\phi^{,\rho}\delta^{\lambda}_{\mu} - \phi_{,\mu}\phi^{,\lambda} + V\delta^{\lambda}_{\mu}\right) = 0\end{aligned} \qquad (5)$$

$$\left[e(1-2\xi T)\phi^{,\mu}\right]_{,\mu} - e\frac{dV}{d\phi} = 0 \qquad (6)$$

As we have seen, this theory is not based on curvatures. This implies that the solutions of (5) and (6) describe a spacetime geometry with metric tensor $g_{\mu\nu} = \eta_{ab}e^a_{\mu}e^b_{\nu}$ may produce images that differ from the standard general relativity. It is interesting to investigate the simplest class of solutions, namely static spacetimes, as an alternative gravity theory.

STATIC SPACETIMES

In this section, we will focus on the static solutions of (5) and (6). As a starting point, consider the ansatz metric given by

$$ds^2 = -N(r)^2dt^2 + K(r)^{-2}dr^2 + R(r)^2 d\Omega^2 \qquad (7)$$

with $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$ is the metric on the submanifold that should permit for constant triplet structures with the torsion scalar $\hat{T}$ belongs to them.

The metric selection as above makes the vielbein of the metric (7) has the form

$$e^a{}_{\mu} = (N(r), K(r)^{-1}, R(r)\hat{e}^{\bar{b}}_i) \qquad (8)$$

Substituting the ansatz (7) to torsion scalar (3) we obtain

$$T = -2K^2\frac{R'}{R}\left[\frac{R'}{R} + 2\frac{N'}{N}\right] \qquad (9)$$

We also get the Ricci scalar as follows

$$\mathrm{R} = -2K^2\left[\frac{K'N'}{KN} + \frac{N''}{N} + 2\frac{K'R'}{KR} + 2\frac{R''}{R} + 2\frac{N'R'}{NR} + \frac{R'^2}{R^2} - \frac{1}{R^2}\right] \qquad (10)$$

MASTER EQUATION

Substituting (8) and (9) into (5) and assume that $\phi = \phi(r)$, we can get several equations as follows

$$\begin{aligned}&2\left(\frac{1}{\kappa_4^2K^2} - 2\xi\phi'^2\right)\left[2\frac{R'K'}{RK} + \frac{R'^2}{R^2} + 2\frac{R''}{R} - \frac{1}{R^2K^2}\right]\\ &-16\xi\phi'\frac{R'}{R}\left(\frac{K'}{K}\phi' + \phi''\right) + \frac{1}{K^2}\left(\phi'^2 + \frac{2V}{K^2}\right) = 0\end{aligned} \qquad (11)$$

$$\begin{aligned}&\left[\frac{R'}{R}\left(\frac{R'}{R} + 2\frac{N'}{N}\right) - \frac{1}{R^2K^2}\right]2\left(2\xi\phi'^2 - \frac{1}{\kappa_4^2K^2}\right)\\ &+\left(\frac{R'}{R}\left(\frac{R'}{R} + \frac{2N'}{N}\right)\right)8\xi\phi'^2 + \frac{1}{K^2}\left(\phi'^2 - \frac{2V}{K^2}\right) = 0\end{aligned} \qquad (12)$$

$$\begin{aligned}&2\left(\frac{1}{\kappa_4^2K^2} - 2\xi\phi'^2\right)\left[\frac{N'}{N}\left(\frac{R'}{R} + \frac{K'}{K}\right) + \frac{R''}{R} + \frac{R'K'}{RK} + \frac{N''}{N}\right]\\ &-8\xi\phi'\left(\frac{R'}{R} + \frac{N'}{N}\right)\left(\frac{K'}{K}\phi' + \phi''\right) + \frac{1}{K^2}\left(\phi'^2 + \frac{2V}{K^2}\right) = 0\end{aligned} \qquad (13)$$

$$\phi'\left(\frac{K'}{K}\phi' + \phi''\right) = 0 \qquad (14)$$

In similar way, equation (6) in this case becomes

$$\left\{\left[1 + 4\xi\left(\frac{K^2R'}{R}\left[\frac{R'}{R} + \frac{2N'}{N}\right]\right)\right]\times NR^2K\phi'\right\}' - \frac{NR^2}{K}\frac{dV}{d\phi} = 0 \qquad (15)$$

One equation in (11)-(14) can be expected to be the constraint of the theory. We choose equation (14) which can be simplified into

$$\phi' = \frac{\nu}{K} \tag{16}$$

where ν is a constant.

Next, while intoducing $\dot{f} = \partial f / \partial \phi$, we define new variables as in [12]

$$Y \equiv y^2, \quad Z \equiv z^2 \tag{17}$$

where x, y, z define as

$$x = \ln R \tag{18}$$

$$y = \frac{\dot{R}}{R} \tag{19}$$

$$z = \frac{\left(RN^2\right)^{\cdot}}{RN^2} \tag{20}$$

So that we can obtain

$$\dot{y} = \frac{1}{2}\frac{dY}{dx} \tag{21}$$

$$\dot{z} = \frac{1}{2}\zeta\sqrt{\frac{Y}{Z}}\frac{dZ}{dx} \tag{22}$$

Then, (11)-(13) become

$$\frac{dY}{dx} + 3Y - \frac{e^{-2x}}{\nu^2} + \frac{\kappa_4^2\left(\nu^2 + 2V\right)}{2\nu^2\left(1 - 2\xi\nu^2\kappa_4^2\right)} = 0 \tag{23}$$

$$\sqrt{YZ} - \frac{\left(2\xi\nu^2\kappa_4^2 - 1\right)e^{-2x}}{\nu^2\left(6\xi\nu^2\kappa_4^2 - 1\right)} + \frac{\kappa_4^2\left(\nu^2 - 2V\right)}{2\nu^2\left(6\xi\nu^2\kappa_4^2 - 1\right)} = 0 \tag{24}$$

$$\zeta\sqrt{\frac{Y}{Z}}\frac{dZ}{dx} + Z + \frac{e^{-2x}}{\nu^2} + \frac{3\kappa_4^2\left(\nu^2 + 2V\right)}{2\nu^2\left(1 - 2\xi\nu^2\kappa_4^2\right)} = 0 \tag{25}$$

NO SOLUTION IN FOUR DIMENSIONS

We consider the exact potential namely sinh-Gordon potential,

$$V(x) = \alpha e^{-2x} + \beta e^{2x} + \gamma \tag{26}$$

with α, β, γ are constants. For the sake of simplicity, consider the following exact solution in the fourth dimension, we can write equation (23) as

$$\frac{dY}{dx} + 3Y + a_1 e^{-2x} + a_2 e^{2x} + a_3 = 0 \tag{27}$$

The solution to this differential equation is

$$Y(x) = -a_1 e^{-2x} - \frac{1}{5}a_2 e^{2x} - \frac{a_3}{3} + a_4 e^{-3x} \tag{28}$$

where a_4 is a constant and

$$a_1 = -\frac{1}{\nu^2} + \frac{\kappa_4^2\alpha}{\nu^2\left(1 - 2\xi\nu^2\kappa_4^2\right)} \tag{29}$$

$$a_2 = \frac{\kappa_4^2\beta}{\nu^2\left(1 - 2\xi\nu^2\kappa_4^2\right)} \tag{30}$$

$$a_3 = \frac{\kappa_4^2\left(\nu^2 + 2\gamma\right)}{2\nu^2\left(1 - 2\xi\nu^2\kappa_4^2\right)}. \tag{31}$$

Then, using the result in (28) and inserting to (24) for $D = 4$, we obtain

$$Z = \frac{\left(b_1 e^{-2x} + b_2 e^{2x} + b_3\right)^2}{\left(-a_1 e^{-2x} - \frac{1}{5}a_2 e^{2x} - \frac{a_3}{3} + a_4 e^{-3x}\right)} \tag{32}$$

where

$$b_1 = \frac{2\xi\nu^2\kappa_4^2 - 1 + \kappa_4^2\alpha}{\nu^2\left(6\xi\nu^2\kappa_4^2 - 1\right)} \tag{33}$$

$$b_2 = \frac{\kappa_4^2\beta}{\nu^2\left(6\xi\nu^2\kappa_4^2 - 1\right)} \tag{34}$$

$$b_3 = \frac{\kappa_4^2\left(2\gamma - \nu^2\right)}{2\nu^2\left(6\xi\nu^2\kappa_4^2 - 1\right)} \tag{35}$$

In the same way, substituting (32) and (27) to (25), we can get

$$\begin{aligned}
&\left(4\zeta b_1 a_1 - 2\zeta a_1 b_1 + b_1 b_1 + c_1 a_1\right)e^{-4x} \\
&+\left(b_2 b_2 + \frac{2}{5}\zeta a_2 b_2 - \frac{4}{5}\zeta b_2 a_2 + \frac{1}{5}c_2 a_2\right)e^{4x} \\
&+\left(4\zeta b_1\frac{a_3}{3} - 2\zeta b_3 a_1 + 2b_1 b_3 + c_1\frac{a_3}{3} + a_1 c_3\right)e^{-2x} \\
&+\left(2b_2 b_3 + \frac{2}{5}\zeta a_2 b_3 - \frac{4}{3}\zeta a_3 b_2 + \frac{a_3}{3}c_2 + \frac{1}{5}a_2 c_3\right)e^{2x} \\
&+\left(\frac{6}{5}\zeta b_1 a_2 - 6\zeta b_2 a_1 + 2b_1 b_2 + b_3 b_3 + \frac{1}{5}c_1 a_2 + c_2 a_1 + \frac{a_3}{3}c_3\right) \\
&+\left(3\zeta a_4 b_1 - 4\zeta b_1 a_4 - c_1 a_4\right)e^{-5x} + \left(3\zeta b_3 a_4 - a_4 c_3\right)e^{-3x} \\
&+\left(4\zeta b_2 a_4 + 3\zeta a_4 b_2 - a_4 c_2\right)e^{-x} = 0
\end{aligned} \tag{36}$$

where

$$c_1 = -\left(\frac{1}{\nu^2} + \frac{3\kappa_4^2\alpha}{\nu^2\left(1 - 2\xi\nu^2\kappa_4^2\right)}\right) \tag{37}$$

$$c_2 = -\frac{3\kappa_4^2\beta}{\nu^2\left(1 - 2\xi\nu^2\kappa_4^2\right)} \tag{38}$$

$$c_3 = -\frac{\left(3\kappa_4^2\nu^2 + 6\kappa_4^2\gamma\right)}{2\nu^2\left(1 - 2\xi\nu^2\kappa_4^2\right)} \tag{39}$$

From (36) we take $a_4 = 0$, and we can obtain that $\beta = 0$, Hence, we have the following equation

$$2\zeta b_1 a_1 + b_1 b_1 + c_1 a_1 = 0 \qquad (40)$$

$$b_3 b_3 + \frac{a_3}{3} c_3 = 0 \qquad (41)$$

$$4\zeta b_1 \frac{a_3}{3} - 2\zeta b_3 a_1 + 2b_1 b_3 + c_1 \frac{a_3}{3} + a_1 c_3 = 0 \qquad (42)$$

Substituting (29), (33), and (37) to (40), we obtain

$$p_1 \alpha^2 + p_2 \alpha + p_3 = 0 \qquad (43)$$

where

$$p_1 = \frac{\kappa_4^4}{\nu^4 \left(6\xi\nu^2\kappa_4^2 - 1\right)^2} - \frac{3\kappa_4^4}{\nu^4 \left(1 - 2\xi\nu^2\kappa_4^2\right)^2} + 2\zeta \frac{\kappa_4^4}{\nu^4 \left(6\xi\nu^2\kappa_4^2 - 1\right)\left(1 - 2\xi\nu^2\kappa_4^2\right)} \qquad (44)$$

$$p_2 = \frac{2\kappa_4^2 \left(2\xi\nu^2\kappa_4^2 - 1\right)}{\nu^4 \left(6\xi\nu^2\kappa_4^2 - 1\right)^2} + \frac{2\kappa_4^2}{\nu^4 \left(1 - 2\xi\nu^2\kappa_4^2\right)} - \frac{\kappa_4^2}{\nu^4 \left(6\xi\nu^2\kappa_4^2 - 1\right)} - \frac{2\zeta\kappa_4^2}{\nu^4 \left(6\xi\nu^2\kappa_4^2 - 1\right)} \qquad (45)$$

$$p_3 = \frac{1}{\nu^4} - \frac{2\xi\nu^2\kappa_4^2 - 1}{\nu^4 \left(6\xi\nu^2\kappa_4^2 - 1\right)} + \frac{\left(2\xi\nu^2\kappa_4^2 - 1\right)^2}{\nu^4 \left(6\xi\nu^2\kappa_4^2 - 1\right)^2} \qquad (46)$$

Solving the quadratic equation (43), we can get

$$\alpha = \frac{-p_2 \pm \sqrt{p_2^{\,2} - 4p_1 p_3}}{2p_1} \qquad (47)$$

In the same way, using (41), we attain

$$\gamma = \frac{2\xi\nu^4\kappa_4^2 \left(3\kappa_4 - 1\right) + \nu^2 \left(1 - \kappa_4\right)}{2\left(1 + \kappa_4\right) - 4\xi\nu^2\kappa_4^2 \left(1 + 3\kappa_4\right)} \qquad (48)$$

Finally, when we substitute (47) and (48) into (42), we can not get consistent results so there is no exact solution for the above potential case.

CONCLUSION

In four dimensions, we discussed scalar-torsion theory with potential turn on. We focused on a class of static spacetimes in which the equations of motion can be reduced to a single non-linear ordinary differential equation known as the master equation. Then, we demonstrate that there are no exact solution of the master equation in the fourth dimension where the constant in (26) satisfies (47)-(48) and $\beta = 0$.

ACKNOWLEDGMENT

The research in this paper is supported by P2MI FMIPA ITB 2022, Riset KK ITB 2022, and Riset PDUPT Kemendikbudristek-ITB 2022.

REFERENCES

[1] R. Aldrovandi and J. G. Pereira, Teleparallel Gravity, Springer Verlag, (2013).

[2] R. Bengochea and R. Ferraro, Dark torsion as the cosmic speed-up, Phys. Rev. D **79** (2009), 124019 [arXiv:0812.1205 [astro-ph]]

[3] T. Wang, Static Solutions with Spherical Symmetry in f(T) Theories, Phys. Rev. D **84** (2011) 024042 [arXiv:1102.4410 [gr-qc]]

[4] C. G. Boehmer, A. Mussa and N. Tamanini, Existence of relativistic stars in f(T) gravity, Class. Quant. Grav. **28** (2011) 245020 [arXiv:1107.4455 [gr-qc]].

[5] R. Ferraro and F. Fiorini, Spherically symmetric static spacetimes in vacuum f(T) gravity, Phys. Rev. D **84** (2011) 083518 [arXiv:1109.4209 [gr-qc]].

[6] H. Abedi and S. Capozziello, Gravitational waves in modified teleparallel theories of gravity, Eur. Phys. J. C **78** (2018) 474.

[7] G. Kofinas, E. Papantonopoulos and E. N. Saridakis, Self-Gravitating Spherically Symmetric Solutions in Scalar-Torsion Theories, Phys. Rev. D **91** (2015) 104034, [arXiv:1501.00365 [gr-qc]]

[8] A. Yaqin and B. E. Gunara, Comment on Self-gravitating spherically symmetric solutions in scalar-torsion theories, Phys. Rev. D **96** (2017) 028501 [arXiv:1706.04710[gr-qc]].