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Einstein-Klein-Gordon System in Higher Dimensional

Mirda Prisma Wijayanto[1*], **Fiki Taufik Akbar Sobar**[1,2], **Bobby Eka Gunara**[1,2]
[1]*Theoretical of High Energy Physics Research Division, Institut Teknologi Bandung*
[2]*Indonesia Center for Theoretical and Mathematical Physics (ICTMP)*

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Abstract

In this present work, we study the Einstein equation coupled with the nonlinear Klein-Gordon equation. We obtain Ricci tensor, scalar curvature, and Einstein equation of the Einstein-Klein-Gordon system in higher dimensional. If we put $D = 4$, our formulations reduce to the four dimensional Einstein-Klein-Gordon system.

Keywords: Einstein equation, Klein-Gordon equation

INTRODUCTION

The Cauchy problem for Einstein equation was first started by Choquet-Bruhat [4] and the local existence of solutions was proved by using the harmonic coordinates [7, 9, 13]. One of the most remarkable discoveries was the global existence of the Einstein vacuum equation by Christodoulou and Kleinermen [6]. Furthermore, the global existence of Einstein equation in closed space-time has also been proven by Andersson and Moncrief [2]. Then, the Einstein-Maxwell-Yang-Mills equation for small initial data has been studied by Friedrich [8].

It is also interesting to study global properties of solutions to various Einstein-matter equations. Global existence with small data for Einstein equation coupled with other materials such as self-gravitating scalar system has been done by Christodoulou [5], and self-gravitating Vlasov-Poisson system has been done by Rein and Rendall [14], both of which assume that space-time is spherically symmetric.

Now, let us consider Einstein equation coupled with nonlinear Klein-Gordon equation. In general relativity, this system provides an important toy model for the study of gravitational collapse which retains relevant dynamical degrees of freedom under the assumption of spherical symmetry. Study of Klein-Gordon equation in the Minkowski space frame has been extensively studied, including by Struwe [1, 10, 11, 15, 16] and the references therein. The study about nonlinear Klein-Gordon equations coupled with Einstein equations in spherical symmetry has been done previously by Malec [12], where Cauchy data is given on spacelike hypersurface, as well as by examining the local existence of the Cauchy problem outside outgoing null hypersurfaces. Moreover, Chae [3] studied the characteristics of the initial value problem of Einstein equation coupled with the nonlinear Klein-Gordon equation on the four dimension. Hereafter, this system is referred to Einstein-Klein-Gordon (EKG) system. In this present works, we study Einstein-Klein-Gordon system in higher dimensional.

FOUR DIMENSIONAL EKG

Let us consider Lorentzian manifold diffeomorphic to R^4. The group *SO(3)* acts as an isometry and the group orbits are the metric spacelike 2-spheres in which the invariants of the group form a timelike curve in the spacetime. Then, we introduce the function

[*] Corresponding author.
E-mail address: mirda.prisma.wijayanto@students.itb.ac.id

$$r = \sqrt{\frac{A}{4\pi}} \quad (1)$$

where A is the area of the 2-sphere. The metric on the 2-sphere is given by [3]

$$ds^2 = r^2 d\Sigma^2 = r^2(d\theta^2 + \sin^2\theta d\Phi^2). \quad (2)$$

We define a new coordinate u such that the metric (2) can be written as

$$ds^2 = -e^{2F}du^2 - 2e^{F+G}dudr + r^2 d\Sigma^2 \quad (3)$$

where the function F dan G tend to 0 as r goes to infinity according to asymptotic flatness. Metric (3) has non-zero components as follows

$$g_{00} = -e^{2F} \quad (4)$$
$$g_{01} = g_{10} = -e^{F+G} \quad (5)$$
$$g_{22} = r^2 \quad (6)$$
$$g_{33} = r^2 \sin^2\theta \quad (7)$$

with the inverse components

$$g^{01} = g^{10} = -e^{-(F-G)} \quad (8)$$
$$g^{11} = e^{-2G} \quad (9)$$
$$g^{22} = \frac{1}{r^2} \quad (10)$$
$$g^{33} = \frac{1}{r^2\sin^2\theta}. \quad (11)$$

We obtain the non-zero Christoffel symbols of metric (3) as follows

$$\Gamma^0_{00} = \partial_0(F+G) - e^{F-G}\partial_1 F \quad (12)$$
$$\Gamma^0_{22} = re^{-(F+G)} \quad (13)$$
$$\Gamma^0_{33} = re^{-(F+G)}\sin^2\theta \quad (14)$$
$$\Gamma^1_{00} = -e^{F-G}\partial_0 G + e^{2(F-G)}\partial_1 F \quad (15)$$
$$\Gamma^1_{01} = \Gamma^1_{10} = e^{F-G}\partial_1 F \quad (16)$$
$$\Gamma^1_{11} = \partial_1(F+G) \quad (17)$$
$$\Gamma^1_{22} = -re^{-(F+G)} \quad (18)$$
$$\Gamma^1_{33} = -re^{-2G}\sin^2\theta \quad (19)$$
$$\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r} \quad (20)$$
$$\Gamma^2_{33} = -\sin\theta\cos\theta \quad (21)$$
$$\Gamma^3_{23} = \Gamma^3_{32} = \cot\theta \quad (22)$$

From the above Christoffel symbols, we obtain Ricci tensor as follows

$$\begin{aligned} R_{00} = &-e^{F-G}\left[\partial_0\partial_1(F+G) + \frac{2}{r}\partial_0 G\right] \\ &+e^{2(F-G)}[(\partial_1 F)^2 + \partial_1(\partial_1 F) - \\ &\partial_1 F\partial_1 G + \tfrac{2}{r}\partial_1 F] \end{aligned} \quad (23)$$

$$\begin{aligned} R_{01} = R_{10} = &-\partial_0\partial_1(F+G) \\ &+e^{F-G}\partial_1(F-G)\partial_1 F + \\ &e^{F-G}\partial_1(\partial_1 F) + \tfrac{2}{r}e^{F-G}\partial_1 F \end{aligned} \quad (24)$$

$$R_{11} = \frac{2}{r}\partial_1(F+G) \quad (25)$$

$$R_{22} = 1 - e^{-2G}(r\partial_1(F-G)+1) \quad (26)$$
$$R_{33} = \sin^2\theta\,[1 - e^{-2G}(r\partial_1(F-G)+1)] \quad (27)$$

Furthermore, we obtain the scalar curvature

$$\begin{aligned} R = &\frac{2}{r^2} + 2e^{-(F+G)}\partial_0\partial_1(F+G) \\ &+e^{-2G}[-2(\partial_1 F)^2 - 2\partial_1(\partial_1 F) \\ &+2(\partial_1 F)(\partial_1 G) - \tfrac{4}{r}\partial_1(F-G) - \tfrac{2}{r^2}]. \end{aligned} \quad (28)$$

The Einstein equation coupled with the nonlinear Klein-Gordon equation fullfiles

$$R_{\mu\nu} = 8\pi\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right), \quad (29)$$

with the energy-momentum tensor

$$\begin{aligned} T_{\mu\nu} = &\partial_\mu\phi\partial_\nu\phi - \tfrac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi \\ &-g_{\mu\nu}V(\phi), \end{aligned} \quad (30)$$

the scalar potential $V(\phi)$, and T is the trace of $T_{\mu\nu}$. Thus, from the matter field equation

$$\nabla_\mu T^{\mu\nu} = 0 \quad (31)$$

we obtain nonlinear Klein-Gordon equation

$$\Box\phi = \frac{\partial V(\phi)}{\partial\phi} \quad (32)$$

where $\Box$ is defined as the d'Alembert operator as follows

$$\begin{aligned} \Box\phi = &-2e^{-F-G}\left(\frac{\partial^2\phi}{\partial u\partial r} + \frac{1}{r}\frac{\partial\phi}{\partial u}\right) \\ &+e^{-2G}\left[\frac{\partial^2\phi}{\partial r^2} + \left(\frac{2}{r} + \frac{\partial}{\partial r}(F-G)\right)\frac{\partial\phi}{\partial r}\right]. \end{aligned} \quad (33)$$

The $\{rr\}$ component of (29) can be written down as

$$\frac{\partial}{\partial r}(F+G) = 4\pi r\left(\frac{\partial\phi}{\partial r}\right)^2. \quad (34)$$

The $\{\theta\theta\}$ or $\{\Phi\Phi\}$ component of (29) can be written down as

$$\frac{\partial}{\partial r}(F-G) + \frac{1}{r}(e^{2G} - 1) = -8\pi r e^{2G}V(\phi). \quad (35)$$

HIGHER DIMENSIONAL EKG

In higher dimensional, we can written down the metric (3) as

$$ds^2 = -e^{2F}du^2 - 2e^{F+G}dudr + r^2 d\Sigma^{d-2} \quad (36)$$

with the non-zero metric components

$$g_{00} = -e^{2F} \quad (37)$$
$$g_{01} = g_{10} = -e^{F+G} \quad (38)$$
$$g_{ij} = r^2\,\hat{\sigma}_{ij} \quad (39)$$

where $\hat{\sigma}_{22} = 1$ and $\hat{\sigma}_{33} = \sin^2\theta$. Then, we have the inverse of the metric components

$$g^{01} = g^{10} = -e^{-(F-G)} \quad (40)$$
$$g^{11} = e^{-2G} \quad (41)$$
$$g^{ij} = \frac{1}{r^2}\hat{\sigma}^{ij} \quad (42)$$

where $\hat{\sigma}^{22} = 1$ and $\hat{\sigma}^{33} = \sin^2\theta$. We obtain the non-zero Christoffel symbols of metric (36) as follows

$$\Gamma^0_{00} = \partial_0(F+G) - e^{F-G}\partial_1 F \quad (43)$$

$$\Gamma^0_{ij} = re^{-(F+G)}\hat{\sigma}_{ij} \quad (44)$$

$$\Gamma^1_{00} = -e^{F-G}\partial_0 G + e^{2(F-G)}\partial_1 F \quad (45)$$

$$\Gamma^1_{01} = \Gamma^1_{10} = e^{F-G}\partial_1 F \quad (46)$$

$$\Gamma^1_{11} = \partial_1(F+G) \quad (47)$$

$$\Gamma^1_{ij} = -re^{-2G}\hat{\sigma}_{ij} \quad (48)$$

$$\Gamma^i_{1j} = \Gamma^i_{j1} = \frac{1}{r}\delta^i_j \quad (49)$$

$$\Gamma^k_{ij} = \hat{\Gamma}^k_{ij}(\hat{\sigma}) \quad (50)$$

From the above Christoffel symbols, we obtain Ricci tensor as follows

$$\begin{aligned} R_{00} = &-e^{F-G}\left[\partial_0\partial_1(F+G) + \frac{(D-2)}{r}\partial_0 G\right] \\ &+e^{2(F-G)}[(\partial_1 F)^2 + \partial_1(\partial_1 F) \\ &-\partial_1 F\partial_1 G + \frac{(D-2)}{r}\partial_1 F] \end{aligned} \quad (51)$$

$$\begin{aligned} R_{01} = R_{10} = &-\partial_0\partial_1(F+G) \\ &+e^{F-G}\partial_1(F-G)\partial_1 F + \\ &e^{F-G}\partial_1(\partial_1 F) + \frac{(D-2)}{r}e^{F-G}\partial_1 F \end{aligned} \quad (52)$$

$$R_{11} = \frac{(D-2)}{r}\partial_1(F+G) \quad (53)$$

$$\begin{aligned} R_{ij} = &\hat{R}_{ij}(\hat{\sigma}) - e^{-2G}[r\partial_1(F-G) \\ &+D-3]\,\hat{\sigma}_{ij} \end{aligned} \quad (54)$$

with

$$\hat{R}_{ij}(\hat{\sigma}) = \partial_k\Gamma^k_{ij} - \partial_j\Gamma^k_{ik} + \Gamma^k_{ij}\Gamma^\lambda_{k\lambda} - \Gamma^k_{i\lambda}\Gamma^\lambda_{kj} \quad (55)$$

where

$$\hat{R}_{22} = 1 \quad (56)$$

$$\hat{R}_{33} = \sin^2\theta \quad (57)$$

If we put $D = 4$, equation (51 – 54) can be reduced to (23 – 27).

Thus, the scalar curvature can be written down as

$$\begin{aligned} R = &\frac{1}{r^2}\hat{R}(\hat{\sigma}) + 2e^{-(F+G)}\partial_0\partial_1(F+G) \\ &+e^{-2G}[-2(\partial_1 F)^2 - 2\partial_1(\partial_1 F) \\ &+2(\partial_1 F)(\partial_1 G) - \frac{2(D-2)}{r}\partial_1(F-G) \\ &-\frac{(D-2)(D-3)}{r^2}], \end{aligned} \quad (58)$$

with

$$\hat{R}(\hat{\sigma}) = \hat{\sigma}^{ij}\hat{R}_{ij}(\hat{\sigma}). \quad (59)$$

In four dimensional spherical coordinates, we have $\hat{R}(\hat{\sigma}) = 2$. Hence, if we put $D = 4$, equation (59) can be reduced to (28).

The Einstein equation coupled with the nonlinear Klein-Gordon equation in higher dimensional fullfiles

$$R_{\mu\nu} = 8\pi\left(T_{\mu\nu} - \frac{1}{(D-2)}g_{\mu\nu}T\right) \quad (60)$$

We obtain nonlinear Klein-Gordon equation correspond to (38) in higher dimensional, with d'Alembert operator as follows

$$\begin{aligned} \Box\phi = &-2e^{-F-G}\left(\frac{\partial^2\phi}{\partial u\partial r} + \frac{(D-2)}{2r}\frac{\partial\phi}{\partial u}\right) \\ &+e^{-2G}\left[\frac{\partial^2\phi}{\partial r^2} + \left(\frac{(D-2}{r} + \frac{\partial}{\partial r}(F-G)\right)\frac{\partial\phi}{\partial r}\right]. \end{aligned} \quad (61)$$

If we put $D = 4$, then (61) reduce to four dimensional d'Alembert operator as (33).

Finally, the $\{rr\}$ component of (60) can be written down as

$$\frac{\partial}{\partial r}(F+G) = \frac{8\pi r}{D-2}\left(\frac{\partial\phi}{\partial r}\right)^2. \quad (62)$$

If we put $D = 4$, then (63) reduce to (34).

The $\{ij\}$ component of (60) can be written as

$$\begin{aligned} &\partial_1(F-G) - \frac{1}{r}[e^{2G}\Lambda(D-2) - (D-3)] = \\ &-\frac{16\pi re^{2G}}{(D-2)}V(\phi) \end{aligned} \quad (63)$$

where $\hat{R}_{ij}(\hat{\sigma}) = \Lambda(D-2)\hat{\sigma}_{ij}$. If we put $D = 4$, then (64) reduce to (35).

CONCLUSION

In this present work, we study the Einstein-Klein-Gordon system in higher dimensional. We give the formulation of Ricci tensor, scalar curvature, and Einstein equation. If we put $D = 4$, our formulations reduce to the four dimensional Einstein-Klein-Gordon system.

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