Indonesian Journal of Physics
Vol 16 No. 2, April 2005

Symmetries of the Maxwell-Harmuth's Equations

Triyanta[1)] *and Gunawati*[2)]
[1)] *Department of Physics, FMIPA, Institut Teknologi Bandung, Bandung*
[2)] *Jurusan Fisika, FMIPA, Universitas Syiah Kuala, Banda Aceh*
E-mail: triyanta@fi.itb.ac.id

Abstract

Since its birth one and a half century ago the Maxwell's equations have been variously reformulated and extended. The set of the Maxwell-Harmuth's equations is an example of the extension of the original Maxwell's equations. Unlike to the original ones the Maxwell-Harmuth's equations have the signal solutions and therefore obey the causality law. In addition the Maxwell-Harmuth's electrodynamics has the U(2) symmetry, in contrast to the U(1) symmetry of the original one. The new theory, however, does not belong to the (non-Abelian) Yang-Mills theory, i.e. the Abelian nature of the original Maxwell theory keeps unchanged. Finally, as it is identical to the free-Maxwell theory the free-Maxwell-Harmuth theory has the rotation, SO(2), symmetry.

Keywords: *Modified Maxwell's equations, Maxwell-Harmuth's equations, causality, signal solutions, symmetries.*

1. Introduction

The Maxwell's equations have been among fundamental laws of physics for more than one and a half century. The equations that govern all electromagnetic phenomena consist of eight equations as the following (in MKSA units)[1)]:

$$\nabla \cdot \vec{D} = \rho, \qquad \nabla \cdot \vec{B} = 0, \tag{1.1a}$$

$$\nabla \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}, \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}. \tag{1.1b}$$

In the above equations $\vec{E}$ and $\vec{H}$ are, respectively, the electric and the magnetic fields, $\vec{D}$ the electric displacement, $\vec{B}$ the magnetic induction, while ρ and $\vec{j}$ are the electric charge and the electric current densities, respectively. The second equation of (1.1a) describes the nonexistence of magnetic monopoles. The electric charge and the electric current densities fulfill the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0. \tag{1.2}$$

The Maxwell's equations (1.1a) and (1.1b) are not the original equations introduced by James Clerk Maxwell. The original ones consist of twenty equations and are written as follows[2,3)]:

$$p' = p + \frac{df}{dt}, \quad q' = q + \frac{dg}{dt}, \quad r' = r + \frac{dh}{dt}, \tag{1.3a}$$

$$\frac{d\gamma}{dy} - \frac{d\beta}{dz} = 4\pi p', \quad \frac{d\alpha}{dz} - \frac{d\gamma}{dx} = 4\pi q', \quad \frac{d\beta}{dx} - \frac{d\alpha}{dy} = 4\pi r', \tag{1.3b}$$

$$\mu\alpha = \frac{dH}{dy} - \frac{dG}{dz}, \quad \mu\beta = \frac{dF}{dz} - \frac{dH}{dx}, \quad \mu\gamma = \frac{dG}{dx} - \frac{dF}{dy}, \tag{1.3c}$$

$$\left.\begin{aligned} P &= \mu\left(\gamma\frac{dy}{dt} - \beta\frac{dz}{dt}\right) - \frac{dF}{dt} - \frac{d\Psi}{dx}, \\ Q &= \mu\left(\alpha\frac{dz}{dt} - \gamma\frac{dx}{dt}\right) - \frac{dG}{dt} - \frac{d\Psi}{dy}, \\ R &= \mu\left(\beta\frac{dx}{dt} - \alpha\frac{dy}{dt}\right) - \frac{dH}{dt} - \frac{d\Psi}{dz}, \end{aligned}\right\} \tag{1.3d}$$

$$P = kf, \quad Q = kg, \quad R = kh, \tag{1.3e}$$

$$P = -\varsigma p, \quad Q = -\varsigma q, \quad R = -\varsigma r, \tag{1.3f}$$

$$e + \frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} = 0, \tag{1.3g}$$

$$\frac{de}{dt} + \frac{dp}{dx} + \frac{dq}{dy} + \frac{dr}{dz} = 0. \tag{1.3h}$$

The twenty quantities involved in the original Maxwell's equations (1.3) are *p, q, r, p', q', r', P, Q, R, f, g, h, F, G, H, e*, α, β, γ, and Ψ. There are also three parameters, namely, μ, *k*, and ς. The meaning of the above quantities and parameters are easily understood through equalizing the equations (1.3) with the equations (1.1) and (1.2). For example, the equations (1.3a) describe the current displacement and together with the equations (1.3b) are equal to the Ampere's law (the first equation of (1.1b)) with α, β, and γ are the components of the magnetic field. Completely, one will find the following meanings. (*p,q,r*) are the electric currents defined by Ampere while (*p',q',r'*) are those proposed by Maxwell and (*f, g, h*) are the associated electric displacements; (*F, G, H*) and ψ are vector and scalar potentials respectively and (*P, Q, R*) are the electric fields. The constants are: μ the magnetic permeability, k=1/ε with ε the electric permitivity, c=1/σ with σ the electric conductivity and *e* charge density. Please note that the original equations (1.3) are written not only in terms of electric and magnetic fields but also in terms of the vector and scalar potentials. The continuity equation is also among the equations (1.3). It turns out that the equations (1.1) are more compact than the equations (1.3). This is perhaps the reason why the equations (1.1), rather than the equations (1.3), are much more familiar as the Maxwell's equations.

Unlike the right hand sides, the left hand sides of the two equations in (1.1a) have a similar form, and analogously for the equations (1.1b). The less symmetrical of the right hand side of the Maxwell's equations leads many physicists to modify the law by adding new terms in such a way that the modified Maxwell's equations are more symmetrical. Section 2 is devoted to review some modified Maxwell's equations. Note, however, that not all modifications are due to symmetry consideration. The Harmuth's modification[4-7] presented in Section 3, for example, was not triggered by a symmetrical consideration but, on the other hand, by disobeying the Maxwell's equations from the causality law. Harmuth showed an example that, in contrast to the solutions of the Maxwell's equations, a signal solution of the Maxwell-Harmuth's equations is in accordance with the causality law. Such an interesting fact leads one to study further the Maxwell-Harmuth's equations. This includes the symmetry properties of the Maxwell-Harmuth's equations given in Section 4. Section 5 is devoted for conclusions.

2. Some Modifications of the Maxwell's Equations

The familiar Maxwell's equations (1.1) describe a mathematical reformulation of the original Maxwell's equations (1.3). Another well-known mathematical reformulation is the covariant formulation of the Maxwell's equations in vacuum:

$$\begin{aligned} &\partial^{\nu} F_{\nu\mu} = J_{\mu}, \\ &\partial_{\nu} F_{\lambda\mu} + \partial_{\lambda} F_{\mu\nu} + \partial_{\mu} F_{\nu\lambda} = 0, \\ &F_{\mu\upsilon} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}. \end{aligned} \qquad (2.1)$$

In the above notation $\partial^{\mu} = \partial/\partial x_{\mu}, \partial_{\mu} = \partial/\partial x^{\mu}$ where the Greek indices have the values 0,1,2,3 while $F_{\mu\nu}$ is the anti-symmetric field strength tensor ($F_{0k}= -E_k$, k=1,2,3; $F_{12}=B_3=-B_z$, $F_{23}=B_1=-B_x$, $F_{31}=B_2=-B_y$) and $A^{\mu} = (\phi, \vec{A})$ is the four-vector potential. Einstein summation has been applied for repeating indices. It turns out that

$$\vec{E} = -\nabla\phi - (1/c)\partial\vec{A}/\partial t, \qquad \vec{B} = \nabla \times \vec{A} \qquad (2.1a)$$

When the electromagnetic fields and their sources are written in the form of quaternion numbers[3,8] $Q=a+ib+jc+kd$ (a, b, c, and d are real numbers and i, j, and k have the properties $i^2=j^2=k^2= -1$, $ij=k=-ji$, $jk=i=-kj$, $ki=j=-ik$) the Maxwell's equations read[3]

$$\Delta H = J + \dot{D} \qquad \Delta E = \psi - \dot{B} \qquad (2.2)$$

with the dots indicates time derivatives while the differential operator Δ, in contrast to the operator ∇ is defined as $\Delta = i\partial/\partial x + j\partial/\partial y + k\partial/\partial z$. In the above, the quaternions describing the electromagnetic fields and their sources are defined as the following:

$$\begin{aligned} &E = iE_x + jE_y + kE_z, \\ &B = iB_x + jB_y + kB_z, \quad D = \varepsilon E, \quad \mu H = B, \\ &J = ij_x + jj_y + kj_z, \quad \varepsilon\psi = \rho. \end{aligned} \qquad (2.3)$$

The Maxwell's equations may also be written in a more symmetrical form by defining new fields $\vec{N} \equiv \vec{H} - \vec{D}$ and $\vec{P} \equiv \varepsilon(\vec{B} + \vec{E})$. (The original idea was proposed by Munera and Guzma[9] with $\vec{N} = \vec{B} - \vec{E}$ and $\vec{P} = \vec{B} + \vec{E}$ but we found that their reformulation in terms of their new fields are inconsistence with the Maxwell's equations). In terms of these fields the equations read

$$\nabla \times \vec{N} = \frac{\partial \vec{P}}{\partial t} + \vec{J}, \quad \nabla \cdot \vec{N} = -\rho,$$

$$\frac{1}{\mu\nu} \nabla \times \vec{P} = -\frac{\partial \vec{N}}{\partial t} + \vec{J}, \quad \nabla \cdot \vec{P} = \rho. \qquad (2.4)$$

The Maxwell's equations applied in some specific configurations are some times nicknamed differently. The Maxwell-Klimontovich's equations[10,11], and their modification[12], for example, are the Maxwell's equations applied in the analysis of high gain free electron lasers. These are just the Maxwell's equations but with very specific expressions of the external sources.

We have shown the Maxwell's equations in some different mathematical appearances. In fact, there are identical to the original equations (1.3). The following are some of more essential modifications of the Maxwell's equations, i.e. there are basically extensions of the Maxwell's equations. The first one is a modification by replacing the partial differentiation with respect to time that appears in the Maxwell's equations (1.1b) by the total derivative[3]:

$$\nabla \times \vec{H} = \frac{d\vec{D}}{dt} + \vec{j} = \frac{\partial \vec{D}}{\partial t} + \vec{v} \cdot \nabla \vec{D} + \vec{j},$$

$$-\nabla \times \vec{E} = \frac{d\vec{B}}{dt} = \frac{\partial \vec{B}}{\partial t} + \vec{v} \cdot \nabla \vec{B}. \qquad (2.5)$$

Thomas Phipps[13] interpreted the velocity that appears in the above equations as the velocity of charges relative to observers. The original Maxwell's equations therefore are just a special case, namely the case $\vec{v} = 0$, of the above equations. The modification leads to changes in, among others, wave equations, expression of electromagnetic fields in terms of potentials, and the continuity equations.

The second modification was due to Dirac[3,14,15]. Dirac proposed a more symmetrical electromagnetic law by defining the magnetic charge density ρ_m and the magnetic current density $\vec{j}_m$ in such a way that the new Maxwell's equations have the form (the electric charge and current density ρ and $\vec{j}$ are written as ρ_e and $\vec{j}_e$ respectively):

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{j}_e, \quad \nabla \cdot \vec{B} = \rho_m,$$

$$-\nabla \times \vec{E} = \frac{\partial \vec{B}}{\partial t} + \vec{j}_m, \quad \nabla \cdot \vec{D} = \rho_e. \qquad (2.6)$$

The above equations give two sets of sources as well as potentials, namely $(\rho_e, \vec{j}_e), (\rho_m, \vec{j}_m)$ and $(\phi_e, \vec{A}_m), (\phi_m, \vec{A}_e)$. Each set of the sources fulfills the continuity equation. To reduce degrees of freedom one may choose a gauge condition, not only for $(\phi_e, \vec{A}_m)$ but also for $(\phi_m, \vec{A}_e)$. By defining ρ_m, Dirac postulated a new monopole, the magnetic monopole. Unfortunately such a monopole has not yet been detected[15].

Other modification was proposed by replacing the original, real, electromagnetic fields by the complex ones and introducing new sources in such a way that all the Maxwell's equations contain source terms[3].

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{j}_R, \quad \nabla \cdot \vec{D} = \rho_R,$$

$$-\nabla \times \vec{E} = \frac{\partial \vec{B}}{\partial t} + i\vec{j}_I, \quad \nabla \cdot \vec{B} = i\rho_I. \tag{2.7}$$

In the above equations $\vec{E} = \vec{E}_R + i\vec{E}_I$, $\vec{B} = \vec{B}_R + i\vec{B}_I$ and analogously for $\vec{D}$ and $\vec{H}$. The above equations give rise the original Maxwell's equations

$$\nabla \times \vec{H}_R = \frac{\partial \vec{D}_R}{\partial t} + \vec{j}_R, \quad \nabla \cdot \vec{D}_R = \rho_R,$$

$$-\nabla \times \vec{E}_R = \frac{\partial \vec{B}_R}{\partial t}, \quad \nabla \cdot \vec{B}_R = 0, \tag{2.8}$$

as well as the additional equations for the imaginary parts of the complex fields:

$$\nabla \times \vec{H}_I = \frac{\partial \vec{D}_I}{\partial t}, \quad \nabla \cdot \vec{D}_I = 0,$$

$$-\nabla \times \vec{E}_I = \frac{\partial \vec{B}_I}{\partial t} + \vec{j}_I, \quad \nabla \cdot \vec{B}_I = \rho_I. \tag{2.9}$$

It is clear that the imaginary number i plays the role for separating the real fields from the imaginary ones.

By choosing appropriate quantities A, C, and Y the two first-order equations below describe the Maxwell's equations or their extensions combined with the Lorentz gauge (summation over repeated α and β indices in the equations below is assumed)[16]:

$$\mu^\alpha \partial^\alpha A = C, \qquad \mu^\beta \partial_\beta C = Y, \tag{2.10}$$

with

$$\mu^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbf{1}, \quad \mu^1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix},$$

$$\mu^2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & i \\ -1 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \quad \mu^3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}. \tag{2.11}$$

In fact, taking

$$A = \begin{pmatrix} \Phi \\ \vec{A} \end{pmatrix}, \quad C = \begin{pmatrix} \Gamma \\ -\vec{E} - i\vec{B} \end{pmatrix}, \quad Y = \begin{pmatrix} \rho + \partial_0 \Gamma \\ \vec{J} - \nabla \Gamma \end{pmatrix}, \tag{2.12}$$

one obtain the Maxwell's equations and the Lorentz gauge as well as the definition of the electromagnetic potentials

$$\begin{aligned} &\partial_0 \Phi + \nabla \cdot \vec{A} = \Gamma, \quad \nabla \times \vec{A} = \vec{B}, \\ &-\nabla \Phi - \partial_0 \vec{A} = \vec{E}, \\ &\nabla \cdot \vec{E} = \rho, \quad \nabla \times \vec{B} - \partial_0 \vec{E} = \vec{J}, \\ &\nabla \cdot \vec{B} = 0, \quad \nabla \times \vec{E} + \partial_0 \vec{B} = 0. \end{aligned} \tag{2.13}$$

When, on the other hand, one takes A and Y complex, i.e.

$$A = \begin{pmatrix} \Phi \\ \vec{A} \end{pmatrix} = \begin{pmatrix} \phi + i\theta \\ \vec{a} + i\vec{b} \end{pmatrix}, \quad C = \begin{pmatrix} \gamma + i\delta \\ -\vec{\varepsilon} - i\vec{\beta} \end{pmatrix}, \quad Y = \begin{pmatrix} \varepsilon + i\beta \\ \vec{j} + i\vec{m} \end{pmatrix}, \tag{2.14}$$

one has the extended Maxwell's equations in the Lorentz gauge:

$$\begin{aligned} &\partial_0 \phi + \nabla \cdot \vec{a} = \gamma, \quad \partial_0 \theta + \nabla \cdot \vec{b} = \delta, \\ &\nabla \times \vec{a} - \nabla \theta - \partial_0 \vec{b} = \vec{\beta}, \\ &-\nabla \times \vec{b} - \nabla \phi - \partial_0 \vec{a} = \vec{\varepsilon}, \\ &\nabla \cdot \vec{\varepsilon} = \varepsilon, \quad \nabla \times \vec{\beta} - \partial_0 \vec{\varepsilon} = \vec{j}, \\ &\nabla \cdot \vec{\beta} = \beta, \quad \nabla \times \vec{\varepsilon} + \partial_0 \vec{\beta} = -\vec{m}. \end{aligned} \tag{2.15}$$

The combination of the equations in (2.10) gives an equation that looks like the Klein-Gordon equation in quantum field theory. This leads Armour[16] to look at the properties of the electromagnetic potentials with respect to the spin-1/2 and spin-1 transformations.

Finally there is another modification that becomes the main object of this study. The modification was proposed by F. Harmuth[4-7] and the corresponding modified Maxwell's equations will be called the Maxwell-Harmuth's equations. We will elaborate these in the remaining sections.

3. Signal Solutions of the Maxwell-Harmuth's Equations

The Maxwell-Harmuth's equations are defined as the following:

$$-\nabla \times \vec{E} = \frac{\partial \vec{B}}{\partial t} + \vec{j}_m, \quad \nabla \cdot \vec{D} = \rho_e,$$

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{j}_e, \tag{3.1a}$$

$$\nabla \cdot \vec{B} = \rho_m \text{ or } \nabla \cdot \vec{B} = 0. \tag{3.1b}$$

The above equations are basically similar to that of Dirac. Harmuth, however, proposed two types of Maxwell equations, namely the types with magnetic monopoles, i.e. the equations (3.1a) plus the first equation of (3.1b), and without magnetic monopoles, i.e. the equations (3.1a) plus the second equation of (3.1b). Experimental development on the existence of the magnetic monopole will judge the right one. At present the second type is in accordance with the experiment. The magnetic current density $\vec{j}_m$ is defined as due to magnetic multipoles, including the magnetic monopole for the first type but excluding it for the second one. Both types fulfill the continuity equations

$$\nabla \cdot \vec{j}_e + \frac{\partial \rho_e}{\partial t} = 0, \quad \nabla \cdot \vec{j}_m + \frac{\partial \rho_m}{\partial t} = 0, \qquad (3.2)$$

with ρ_m is taken to be zero for the second type.

Writing the electric and magnetic current densities in terms of the electric and magnetic fields, $\vec{j}_e = \sigma\vec{E}$, $\vec{j}_m = s\vec{H}$ (σ is the electric conductivity and s is the magnetic conductivity), the equations (3.1) give the following differential equations:

$$\left(\nabla^2 - \mu\varepsilon\frac{\partial^2}{\partial t^2} - (\mu\sigma + \varepsilon s)\frac{\partial}{\partial t} - s\sigma\right)\begin{pmatrix}\vec{E}\\ \vec{H}\end{pmatrix} = 0. \qquad (3.3)$$

The divergence-less property of the magnetic field in the original Maxwell's equations leads one to propose the vector potential. There are no vector potentials associated with the electric field as in general the electric field does not have the divergence-less property. In fact there are scalar potential associated with the field. In non-static cases, where the electric and magnetic fields depends each other, not only the (electric) scalar potential but also the (magnetic) vector potential define the electric field. Since the Maxwell-Harmuth's equations are more symmetrical the relationship between the potentials and the electromagnetic fields for these modified Maxwell's fields naturally should also be more symmetrical. The next section will show this.

The importance of the Maxwell-Harmuth's equations is that, unlike the original Maxwell's equations, the signal solutions of these modified equations fulfill the causality law. To see this let us follow the reference of[7]. Consider, for simplicity, the TEM (transverse electromagnetic) waves propagating along the y axis. Putting $E_x=E_z=E$ and $H_x=H_z=H$ one obtains the differential equations for the Maxwell-Harmuth's TEM waves:

$$\frac{\partial E}{\partial y} + \mu\frac{\partial H}{\partial t} + sH = 0,$$

$$\frac{\partial H}{\partial y} + \varepsilon\frac{\partial E}{\partial t} + \sigma E = 0. \qquad (3.4)$$

The original Maxwell's TEM waves, on the other hand, understandably fulfill the above equations with s=0:

$$\frac{\partial E}{\partial y} + \mu\frac{\partial H}{\partial t} = 0, \quad \frac{\partial H}{\partial y} + \varepsilon\frac{\partial E}{\partial t} + \sigma E = 0, \qquad (3.5)$$

and these lead to the differential equation for the Maxwell's electric fields

$$\frac{\partial^2 E}{\partial y^2} - \mu\varepsilon\frac{\partial^2 E}{\partial t^2} - \mu\sigma\frac{\partial E}{\partial t} = 0\,. \qquad (3.6)$$

Its solutions give rise the corresponding associated magnetic fields through

$$H(y,t) = -\frac{1}{\mu}\int\frac{\partial E}{\partial y}dt + H_t(y), \qquad (3.7)$$

$$H(y,t) = -\int\left(\varepsilon\frac{\partial E}{\partial t} + \sigma E\right)dy + H_y(t), \qquad (3.8)$$

where $H_t(y)$ and $H_y(t)$ are integration constants. Analogously, one may obtain the differential equation for H and the integral equations for its associated electric fields. The TEM Maxwell-Harmuth's electric fields, on the other hand, obey the differential equation

$$\frac{\partial^2 E}{\partial y^2} - \mu\varepsilon\frac{\partial^2 E}{\partial t^2} - (\mu\sigma + \varepsilon s)\frac{\partial E}{\partial t} - s\sigma E = 0. \qquad (3.9)$$

with its solutions give the associated magnetic fields through

$$H(y,t) = e^{-st/\mu}\left(-\frac{1}{\mu}\int\frac{\partial E}{\partial y}e^{st/\mu}\,dt + H_t(y)\right), \qquad (3.10)$$

$$H(y,t) = -\int\left(\varepsilon\frac{\partial E}{\partial t} + \sigma E\right)dy + H_y(t). \qquad (3.11)$$

The Maxwell's equations give no problem with periodic or analytic solutions. However such solutions are unrealistic as experimentally periodic waves cannot be produced. The electromagnetic wave can only be produced in the form of signal, thus non periodic, i.e. the wave that before a certain time, say at t=0, has a zero amplitude. An electric field signal fulfilling (3.6) will not guaranty the existence of its associated magnetic field fulfilling (3.7) or (3.8). Let us consider the electric field with its initial and boundary conditions

$$E(\zeta{>}0,0) = 0, \quad E(0,\theta) = E_0 S(\theta) = \begin{cases} 0 & \theta < 0 \\ E_0 & \theta \geq 0 \end{cases} \qquad (3.12)$$

In the above,

$$\theta = \frac{st}{2\mu}, \qquad \varsigma = \frac{1}{2}sy\sqrt{\frac{\varepsilon}{\mu}} \qquad (3.13)$$

are, respectively, the normalized time and space coordinates. After a lengthy calculation we have the solution of the equation (3.9) with s is set to vanish at the end of the calculation:

$$E_E = \begin{cases} 0 & \theta < \varsigma \\ E_0\left[1 - e^{-\theta}\int_0^{\varsigma}\left(\frac{\theta I_1\left(\sqrt{\theta^2-\eta^2}\right)}{\left(\theta^2\eta^2\right)^{1/2}} + I_0\left(\sqrt{\theta^2-\eta^2}\right)\right)d\eta\right] & \theta > \varsigma \end{cases} \qquad (3.14)$$

where I_0 and I_1 are the modified Bessel functions of the first kind. When the parameter s in the equation (3.9) is taken to be zero from the beginning of the calculation one has the solution of the original Maxwell's equation as follows:

$$E_E = \begin{cases} 0 & \theta < \varsigma \\ E_0\left(e^{-\varsigma} + \varsigma\int_{\varsigma}^{0}\frac{I_1\left(\sqrt{\eta^2-\varsigma^2}\right)e^{-\eta}}{\left(\eta^2-\varsigma^2\right)^{1/2}}d\eta\right) & \theta > \varsigma \end{cases} \qquad (3.15)$$

Even though they look very different, the plots of the equations (3.14) and (3.15) have a similar form (see Fig 1). In Fig.1, solid lines are plots for the equation (3.14) while the star lines for the equation (3.15). When we examine the plots more carefully there are in fact slightly different. In the vicinity of ζ=θ the plot of the equation (3.15) is a straight line while for the equation (3.14) it is (small) curly like a damped oscillator which then changes into a straight line as θ moves away from θ=ζ.

The associated magnetic fields, that are the magnetic fields associated with the above electric fields, are obtained by inserting the solution (3.14) into the equations (3.7) and (3.8) for the Maxwell-Harmuth's equations and the solution (3.15) into the equations (3.10) and (3.11) for the original Maxwell's equations. A very long calculation ends up with the solutions[7]

$$H_E(\varsigma,\theta)=\frac{E_0}{Z}\left[-2\zeta+I'_{E1}(\zeta,\theta)-I_{E2}(\zeta,\theta)\right] \tag{3.16}$$
$$+H_\theta(\varsigma),$$

for the Maxwell-Harmuth's equations where

$$I'_{E1}(\varsigma,\theta)$$
$$=\frac{2}{\pi d}-\frac{2}{\pi}e^{-\theta}\left\{\int_d^1\left(\frac{ch[(1-\eta^2)^{1/2}\theta]}{\eta^2}d\eta+\frac{sh[(1-\eta^2)^{1/2}\theta]}{(1-\eta^2)^{1/2}\eta^2}\right)d\eta+\right. \tag{3.17}$$
$$-\int_0^1\left(ch[(1-\eta^2)^{1/2}\theta]+\frac{(2-\eta^2)sh[(1-\eta^2)^{1/2}\theta]}{2(1-\eta^2)^{1/2}}\right)\left(\frac{\sin\eta\varsigma/2}{\eta/2}\right)^2d\eta+$$
$$\left.\int_0^1\frac{sh[(1-\eta^2)^{1/2}\theta]}{(1-\eta^2)^{1/2}}d\eta\right\},$$

$$I_{E2}(\varsigma,\theta)$$
$$=\frac{4}{\pi}e^{-\theta}\int_1^\infty\left(\cos[(\eta^2-1)^{1/2}\theta]+\frac{(2-\eta^2)\sin[(\eta^2-1)^{1/2}\theta]}{2(\eta^2-1)^{1/2}}\right) \tag{3.18}$$
$$\frac{\cos\eta\varsigma}{\eta^2}d$$

and

$$H_E(\zeta,\theta)=-\frac{4E_0}{\pi Z}e^{-\theta}\times$$
$$\left\{\int_0^1\left(ch[(1-\eta^2)^{1/2}\theta]+\frac{(2-\eta^2)sh[(1-\eta^2)^{1/2}\theta]}{2(1-\eta^2)^{1/2}}\right)\right. \tag{3.19}$$
$$\frac{\cos(\eta\varsigma)}{\eta^2}d\eta$$

for the original Maxwell's equations. In the above $Z=\sqrt{\mu/\varepsilon}$, E_0 is a constant, $H_\theta(\varsigma)$ is a constant of integration, and d<<1. Further analysis shows that the solution (3.16) is finite while the solution (3.19) diverges. We conclude that the original Maxwell's equations do not give signal solutions while the Maxwell-Harmuth's equations do. Thus the Maxwell-Harmuth's equations resolve signal problems found in the original Maxwell's equations.

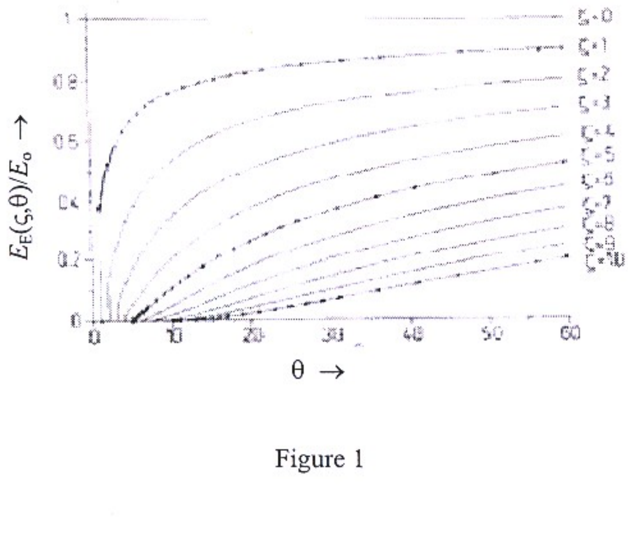

Figure 1

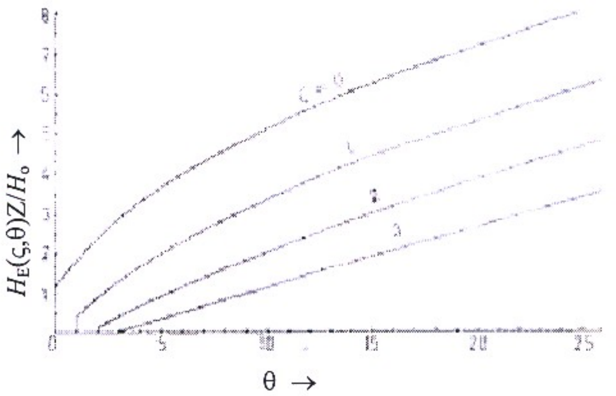

Figure 2

4. Symmetries

Symmetries of a physical system are usually associated with the invariance of the Lagrangian of the system with respect to a set of transformations. Thus to see the symmetry of the ordinary Maxwell's system as well as the Maxwell-Harmuth's system, or other physical systems one should look at their corresponding Lagrangians. The source-free Lagrangian for the ordinary (microscopic) Maxwell's equations read

$$L_{OA}=-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu},\quad F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu. \tag{4.1}$$

In the above, Greek indices μ, ν have the values 0,1,2, and 3, $A^\mu=(A^0=\phi_e,\vec{A}=\vec{A}_m)$, raising and lowering indices are carried out through the Minskowski metric, and summation over repeated indices, one is lower and the other is upper index is understood. ϕ_e is the scalar potential while $\vec{A}_m$ is the vector potential. The index m (magnetic) shows that the potential $\vec{A}_m$ defines the magnetic field while the index e (electric) describes the dependence of the electric field on the potential ϕ_e (in non static cases, however, the electric field also depends on the vector potential). Insertion the above Lagrangian plus the source term $J_\mu A^\mu$ into the Euler-Lagrange equations gives the ordinary Maxwell's equations (in vacuum). It turns out that the Lagrangian (4.1) is invariant under the gauge transformation

$$A_\mu\rightarrow A'_\mu=A_\mu+\partial_\mu\Lambda \tag{4.2}$$

with Λ is some real function of space-time coordinates. As it is proportional to (E^2-B^2) the Lagrangian is not invariant under the two dimensional rotation

$$\begin{pmatrix}\vec{E}\\\vec{B}\end{pmatrix}\rightarrow\begin{pmatrix}\vec{E}'\\\vec{B}'\end{pmatrix}=\begin{pmatrix}\cos\alpha & \sin\alpha\\-\sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\vec{E}\\\vec{B}\end{pmatrix} \tag{4.3}$$

as well as the duality transformation

$$\vec{E}\rightarrow\vec{E}'=\vec{B},\qquad \vec{B}\rightarrow\vec{B}'=-\vec{E}. \tag{4.4}$$

The later transformation is just a special case of the rotation, i.e. the case with $\alpha=90^o$. Since it is proportional to (E^2+B^2), the energy density, unlike the Lagrangian, is invariant under the rotation and the duality transformation. Like the Lagrangian, the energy density is invariant under the gauge transformation. The corresponding Maxwell's equations, on the other hand, are gauge invariant but are not invariant under the group of rotation as well as the duality transformation. Thus the symmetry properties of the Maxwell's equations and their Lagrangian are the same. Note that the source-free Maxwell's equations have the rotation, the duality, as well as the gauge symmetries.

Under the gauge formulation, the (fermion) electrodynamics has the Lagrangian of the form (excluding the source terms):

$$L(\psi,\overline{\psi},A)=\overline{\psi}[i\gamma^{\mu}(\partial_{\mu}-ieA_{\mu}) - m]\psi-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \quad (4.5)$$

In the above, e is the charge of the fermion particle represented by the field ψ. It has the $U(1)$ local gauge symmetry

$$\psi'(x)=e^{ie\Lambda(x)}\psi(x),\qquad A^{\mu}{}'(x)=A^{\mu}(x)+\partial^{\mu}\Lambda(x). \quad (4.6)$$

It is clear that the electromagnetic fields described by the Lagrangian (4.1) is invariant under the above transformations. In this sense we say the electromagnetic fields have the $U(1)$ local gauge symmetry.

All the above description is well known. Let us now expand the Lagrangian (4.1) by introducing two sets of field strength tensors $F^{1\mu\nu}$ dan $F^{2\mu\nu}$:

$$L_A=-\tfrac{1}{4}F^{1}{}_{\mu\nu}F^{1\mu\nu}-\tfrac{1}{4}F^{2}{}_{\mu\nu}F^{2\mu\nu} \quad (4.7)$$

with

$$F^{1\mu\nu}=\partial^{\mu}A^{1\nu}-\partial^{\nu}A^{1\mu},\qquad F^{2\mu\nu}=\partial^{\mu}A^{2\nu}-\partial^{\nu}A^{2\mu}. \quad (4.8)$$

Since each term of this new Lagrangian has the $U(1)$ symmetry, the whole Lagrangian has, accordingly, the $U(1)\otimes U(1)$ symmetry. Let us now associate the potentials $A^{i\mu}$ with the fermion fields ψ_i with their masses m_i, i=1,2. Each pair $(A^{i\mu},\psi_i)$ has the Lagrangian of the form (4.5), and has the $U(1)$ symmetry. The total free Lagrangian for this system is therefore

$$L(\psi,\overline{\psi},A)=\sum_{i=1}^{2}\left(\overline{\psi}_i[i\gamma^{\mu}(\partial_{\mu}-ieA^{i}{}_{\mu})-m_i]\psi_i-\tfrac{1}{4}F^{i}{}_{\mu\nu}F^{i\mu\nu}\right) \quad (4.9)$$

It has the $U(1)\otimes U(1)$ symmetry:

$$\psi_1'(x)=e^{ie\Lambda_1(x)}\psi_1(x),\ A^{1\mu}{}'(x)=A^{1\mu}(x)+\partial^{\mu}\Lambda^{1}(x),\qquad \psi_2'(x)=e^{ie\Lambda_2(x)}\psi_2(x),\ A^{2\mu}{}'(x)=A^{2\mu}(x)+\partial^{\mu}\Lambda^{2}(x). \quad (4.10)$$

One may define a fermion doublet ψ, an electromagnetic field W^{μ}, a differential operator matrix D^{μ}, and a mass matrix M, as the following:

$$\psi=\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix},\qquad W^{\mu}=\begin{pmatrix}A^{1\mu} & 0\\ 0 & A^{2\mu}\end{pmatrix},$$
$$D^{\mu}=\begin{pmatrix}\partial^{\mu}-ieA^{1\mu} & 0\\ 0 & \partial^{\mu}-ieA^{2\mu}\end{pmatrix}, \quad (4.11)$$
$$M=\begin{pmatrix}m_1 & 0\\ 0 & m_2\end{pmatrix}.$$

Accordingly the Lagrangian (4.9) can be written as

$$L(\psi,\overline{\psi},A)=\overline{\psi}[i\gamma_{\mu}D^{\mu}-M]\psi-\tfrac{1}{4}Trf_{\mu\nu}f^{\mu\nu},\qquad f^{\mu\nu}=D^{\mu}W^{\nu}-D^{\nu}W^{\mu}, \quad (4.12)$$

while the transformation (4.10) can be written as

$$\psi'=U\psi,\qquad W'^{\mu}=UW^{\mu}U^{-1}-\tfrac{i}{e}(\partial^{\mu}U)U^{-1},$$
$$U=\begin{pmatrix}\exp[ie\Lambda_1(x)] & 0\\ 0 & \exp[ie\Lambda_2(x)]\end{pmatrix}. \quad (4.13)$$

The diagonal form of D^{μ} and W^{μ} enables one to replace D^{μ} in $f^{\mu\nu}$ (c.f. equation (4.12)) by ∂^{μ}. Since U is a 2x2 unitary matrix the above transformation is a two dimensional unitary transformation. Accordingly the system described by the Lagrangian density (4.9) or (4.12) has a $U(2)$ symmetry. In fact, $U(1)$x$U(1)$ and the above $U(2)$ are isomorphic to each other (Please be careful not to extend this isomorphism to the more general two dimensional unitary symmetry; the above group is just a subgroup of the general two dimensional unitary group). In a special case, i.e. the case where Λ_1= $-\Lambda_2$, the above symmetry reduces to an $SU(2)$ symmetry. It is obvious that the generators of the above transformation are $X_1=(\sigma_0+\sigma_3)/2$ and $X_2=(\sigma_0-\sigma_3)/2$. σ_0 is the 2x2 identity matrix while σ_3 is the third Pauli Matrix. Please note that the equations (4.12) and (4.13) remind us to the non-Abelian gauge, such as the quantum chromodynamics (QCD). The system considered, however, is Abelian since the two different vector (photon) fields do not interact each other. The Abelian nature of the system can also be seen through the vanishing of the commutator $[X_1, X_2]$.

We have restricted our consideration to the (special) $U(2)$ and $SU(2)$ symmetries related to the transformation matrix U given in the equation (4.13). Of course this transformation matrix is just a special case of the general form of $U(2)$ and $SU(2)$ matrices. Accordingly one should not generalize the above conclusions, to the general unitary transformations.

Let us now go back to the Lagrangian (4.7) and define $(\vec{E}_e,\vec{B}_m)$ and $(\vec{E}_m,\vec{B}_e)$, respectively as the electromagnetic fields corresponding to the field strengths $F^{1\mu\nu}$ and $F^{2\mu\nu}$, $A^{1\mu}=(\phi_e,\vec{A}_m)$ and $A^{2\mu}=(\phi_m,\vec{A}_e)$ as the potentials, and $J^{1\mu}=(\rho_e,\vec{J}_e)$ and $J^{2\mu}=(\rho_m,\vec{J}_m)$ as the sources. The Euler-Lagrange equations for the Lagrangian (4.7) together with the source term $L_S=J^{1\mu}A^{1}{}_{\mu}+J^{2\mu}A^{2}{}_{\mu}$ give rise two sets of the microscopic Maxwell's equations:

$$\nabla \cdot \vec{E}_e = \rho_e,\ \nabla \cdot \vec{B}_m = 0,$$
$$\nabla \times \vec{B}_m = \vec{j}_e + \frac{\partial \vec{E}_e}{\partial t},\ \nabla \times \vec{E}_e = -\frac{\partial \vec{B}_m}{\partial t}, \quad (4.14)$$

$$\nabla \cdot \vec{E}_m = \rho_m,\ \nabla \cdot \vec{B}_e = 0,$$
$$\nabla \times \vec{B}_e = \vec{j}_m + \frac{\partial \vec{E}_m}{\partial t},\ \nabla \times \vec{E}_m = -\frac{\partial \vec{B}_e}{\partial t}. \quad (4.15)$$

The fields and the potentials are related one to another through

$$\vec{E}_e = -\nabla\phi_e - \frac{\partial \vec{A}_m}{\partial t},\quad \vec{B}_m = \nabla \times \vec{A}_m,$$
$$\vec{E}_m = -\nabla\phi_m - \frac{\partial \vec{A}_e}{\partial t},\quad \vec{B}_e = \nabla \times \vec{A}_e. \quad (4.16)$$

Let us now define new electromagnetic fields:

$$\vec{E} = \vec{E}_e - \vec{B}_e,\quad \vec{B} = \vec{E}_m + \vec{B}_m. \quad (4.17)$$

This gives

$$\vec{E} = -\nabla\phi_e - \frac{\partial \vec{A}_m}{\partial t} - \nabla \times \vec{A}_e,$$
$$\vec{B} = -\nabla\phi_m - \frac{\partial \vec{A}_e}{\partial t} + \nabla \times \vec{A}_m. \quad (4.18)$$

Now operating Div and Curl to the new fields and recalling the equations (4.14) and (4.15) one has

$$\nabla \cdot \vec{E} = \nabla \cdot \vec{E}_e - \nabla \cdot \vec{B}_e = \rho_e,$$
$$\nabla \cdot \vec{B} = \nabla \cdot \vec{B}_m + \nabla \cdot \vec{E}_m = \rho_m,$$
$$\nabla \times \vec{E} = \nabla \times \vec{E}_e - \nabla \times \vec{B}_e = -\vec{J}_m - \frac{1}{c}\frac{\partial \vec{B}}{\partial t}, \quad (4.19)$$
$$\nabla \times \vec{B} = \nabla \times \vec{B}_m + \nabla \times \vec{E}_m = \vec{J}_e + \frac{1}{c}\frac{\partial \vec{E}}{\partial t}.$$

It turns out that the equations (4.19) are the microscopic Maxwell-Harmuth's equations (3.1). Thus the Lagrangian (4.7) describes the Maxwell-Harmuth's fields. Accordingly, in contrast to the original Maxwell's fields that has a $U(1)$ symmetry, the Maxwell-Harmuth's electromagnetics has the $U(2)$ symmetry.

In the source-free case the Maxwell-Harmuth's equations are identical to the ordinary source-free Maxwell's equations. Therefore the source-free Maxwell-Harmuth's equations are invariant under the gauge, the rotation, and the duality transformations. For the rotation, there are two "vectors", i.e. the pairs $(\vec{E}_e, \vec{B}_m)$ and $(\vec{E}_m, \vec{B}_e)$, that are rotated with the same angle α. There are also two pairs of duality transformations, namely $(\vec{E}_e, \vec{B}_m) \to (\vec{B}_m, -\vec{E}_e)$, $(\vec{E}_m, \vec{B}_e) \to (\vec{B}_e, -\vec{E}_m)$. Thus, the set of the source free Maxwell-Harmuth's equations has an $SO(2)$ symmetry.

5. Conclusions

The Maxwell's equations are among the set of equations that play a central role in physics. Accordingly, many physicists from different generations have been paying very much attention to the equations since their birth, one and a half century ago, until recently. Various mathematical reformulations as well as extensions or modifications of the equations have been proposed, including a modification called the Maxwell-Harmuth's equations. Here Harmuth modified the Maxwell's equations by introducing the magnetic current density. The introduction is based on the fact that magnetic multipoles (magnetic dipoles, quadrupoles, etc) do exist. The inclusion or exclusion of the magnetic monopole within the multipoles depends on the experimental results. The importance consequence of the Harmuth's proposal, in contrast to the original Maxwell's equations, is the possibility to obtain signal solutions, showing the conformity of the new equations to the causality law. Another important property of the Maxwell-Harmuth theory is its symmetries. We show that the theory has a $U(2)$, or $SU(2)$, symmetry. Unlike the $SU(2)$ Yang-Mills fields which are non-Abelian, the Maxwell-Harmuth fields are Abelian. Thus the introduction of the magnetic current density has the effect of symmetry changes, from $U(1)$ to $U(2)$, but it keeps the Abelian property unchanged. Please remind that all conclusions related to the unitary symmetry in this paper is restricted to the special form of the unitary transformation given by the equation (4.13). Generalization to arbitrary unitary transformations is not valid. Finally, in addition to unitary symmetry, the free-Maxwell-Harmuth's fields have the rotation, $SO(2)$, symmetry.

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