An Application of the General Relativity-type Inversion Formulae: the Derivation of the Fock-Schwinger-Friedmann-Robertson-Walker Christoffel Symbol

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Abstract

Similar relationship between Christoffel symbols and Riemann tensors and between gauge potentials and gauge field strengths enable us to treat the Christoffel symbols just like the gauge fields. Restricting the Christoffel symbol by the Fock-Schwinger (FS) condition one obtains the so-call inversion formulae in general relativity theory. These formulae may be utilized for various applications, including the Christoffel symbol-free formulation of the general relativity and calculating the so-called Fock-Schwinger Christoffel symbols for certain Riemann tensors. Here we will derive the Fock-Schwinger-Friedmann-Robertson-Walker (FSFRW) Christoffel symbols.

Keywords: Fock-Schwinger gauge, Inversion formulae

1. Introduction

Gauge theory plays important roles in describing fundamental interactions. The interactions between fundamental particles are mediated by the gauge fields that have some specific features, including the gauge freedom. This gauge freedom enables one to restrict a certain condition, the so-called gauge condition, to the gauge fields. The covariant Lorentz gauge condition has been widely applied since it leads, among others, to simple wave equations of the electromagnetic potentials and it enables one to formulate perturbation theory fully in momentum space. Various gauge conditions, other than the Lorentz one, have been introduced by physicists for various reasons or applications. One of them is the so-called the FS¹⁾ gauge or the Poincare gauge¹. Most people interested in the FS gauge are due to the appearance of the inversion formulae²⁽¹⁾

$$A^\mu(x) = -\int_0^1 d\lambda \lambda x_\nu F^{\mu\nu}(\lambda x). \quad (1)$$

In the above, A^μ are the gauge fields and $F^{\mu\nu}$ are the corresponding field strengths. Their original relationship is defined through

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu - ig[A^\mu, A^\nu] \quad (2)$$

with g is a constant. The inversion formulae (1) only exist when the gauge fields are restricted by the FS gauge condition, $x_\mu A^\mu = 0$. The Bianchi identities

$$\begin{aligned} D_\alpha F_{\beta\gamma} + D_\beta F_{\gamma\alpha} + D_\gamma F_{\alpha\beta} &= 0; \\ D_\alpha F_{\beta\gamma} &= \partial_\alpha F_{\beta\gamma} - ig[A_\alpha, F_{\beta\gamma}] \end{aligned} \quad (3)$$

are the necessary and sufficient conditions for A^μ and $F^{\mu\nu}$ in the inversion formulae (1) to be, respectively,

the gauge fields and the corresponding field strengths³⁾. The similar form between the field strengths (2) and the Riemann tensors in the general theory of relativity enables us to have inversion formulae in the general theory of relativity. These formulae will be shown in the next section. By the use of the formulae we can rewrite some expressions in general relativity, including the covariant derivatives and the geodesic equation, not in terms of Christoffel symbols, but rather, in Riemann tensors, leading to Christoffel symbol-free formulations. This can be seen in Section 3. In Section 4 we derive the Christoffel symbol fulfilling the Fock-Schwinger gauge where the Friedmann –Robertson-Walker Riemannian tensor is chosen. The last section is devoted to conclusions.

2. The inversion formulae in general relativity

Riemannian tensors can be written in the form similar to that of the gauge field strengths (2) as follows,

$$B_{\mu\nu} = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu + [\Gamma_\mu, \Gamma_\nu]. \quad (4)$$

In the above, $B_{\mu\nu}$ and Γ_μ are matrices with the Riemann tensors $R^\alpha_{\beta\mu\nu}$ and the Christoffel symbols $\Gamma^\alpha_{\mu\beta}$, respectively, as their α -rows and β -columns elements. Similar form between (2) and (4) directs one to treat the Christoffel symbols $\Gamma^\alpha_{\mu\beta}$ similar to the gauge fields A_μ . That is, restricting the Christoffel symbols by the FS condition $x^\mu \Gamma^\alpha_{\mu\beta} = 0$ one may obtain the inversion formulae in the general relativity theory⁴⁾,

$$\Gamma^\alpha_{FS\mu\beta}(x) = -\int_0^1 d\lambda \lambda x^\nu R^\alpha_{\beta\mu\nu}(\lambda x). \quad (5)$$

Note that the index FS in the left-hand side of the above equation is added to remind us that the corresponding Christoffel symbol fulfills the inversion

¹ The name Poincare gauge is usually adopted in papers on the general relativity theory, see for example²⁾

² Einstein summation convention is assumed throughout the paper.

formulae. Analogy to the gauge field case, the Bianchi identities in the general theory of relativity⁴⁾,

$$\nabla_\alpha R^\mu{}_{\nu\beta\gamma}(x) + \nabla_\beta R^\mu{}_{\nu\gamma\alpha}(x) + \nabla_\gamma R^\mu{}_{\nu\alpha\beta}(x) = 0, \quad (6)$$

(∇_α is the covariant derivative) become the necessary and sufficient conditions for the inversion formulae (5) to be the relationship between the Christoffel symbols and the Riemann tensors⁴⁾. Other expressions of the inversion formulae may be proven to be⁵⁾:

$$\Gamma_{FS\alpha\beta}^\mu(x) = (1+x\cdot\partial)^{-1} x^\nu R^\mu{}_{\alpha\nu\beta}(x), \quad (7)$$

and

$$\Gamma_{FS\mu}(x) = \sum_{n=0}^{\infty} \frac{1}{n!(n+2)} x^\nu x^{\alpha_1} x^{\alpha_2} \dots x^{\alpha_n} \nabla_{\alpha_1} \nabla_{\alpha_2} \dots \nabla_{\alpha_n} B_{\mu\nu}(0)$$

$(1+x\cdot\partial)^{-1}$ in (7) is an operator whose inverse is $(1+x\cdot\partial)=(1+x^\mu\partial_\mu)$.

3. Christoffel symbol-free formulations

The inversion formulae enable us to eliminate Christoffel symbols. Accordingly we can formulate the general relativity theory without the appearance of the Christoffel symbols. Expressions containing Christoffel symbols are the covariant derivatives, for examples covariant derivatives over tensors up to second rank,

$$\nabla_\alpha V^\mu = \partial_\alpha V^\mu + \Gamma_{\alpha\nu}^\mu V^\nu \quad (9a)$$

$$\nabla_\alpha V_\mu = \partial_\alpha V_\mu - \Gamma_{\alpha\mu}^\nu V_\nu \quad (9b)$$

$$\nabla_\alpha V^{\mu\nu} = \partial_\alpha V^{\mu\nu} + \Gamma_{\alpha\lambda}^\mu V^{\lambda\nu} + \Gamma_{\alpha\lambda}^\nu V^{\mu\lambda} \quad (9c)$$

$$\nabla_\alpha V_\nu^\mu = \partial_\alpha V_\nu^\mu + \Gamma_{\alpha\lambda}^\mu V_\nu^\lambda - \Gamma_{\alpha\nu}^\lambda V_\lambda^\mu \quad (9d)$$

$$\nabla_\alpha V_{\mu\nu} = \partial_\alpha V_{\mu\nu} - \Gamma_{\alpha\mu}^\lambda V_{\lambda\nu} - \Gamma_{\alpha\nu}^\lambda V_{\mu\lambda} \quad (9e)$$

and the geodesic equations⁶⁾ (the dot indicates a differentiation with respect to $x^0=ct$)

$$\ddot{x}^k + (\Gamma_{\alpha\beta}^k - \Gamma_{\alpha\beta}^0 \dot{x}^k) \dot{x}^\alpha \dot{x}^\beta = 0. \quad (10)$$

Taking into account Christoffel symbols fulfilling the FS condition, i.e. replacing the Christoffel symbols $\Gamma_{\alpha\beta}^\rho$ in the equations (9) and (10) by their FS counterpart $\Gamma_{FS\alpha\beta}^\rho$, the expressions (9) and (10) can be written, using the inversion formulae (7), in the forms of the Christoffel symbol-free covariant derivatives

$$\nabla_\alpha V^\mu = \partial_\alpha V^\mu - (1+x\cdot\partial)^{-1} x^\sigma R^\mu{}_{\alpha\nu\sigma} V^\nu \quad (11a)$$

$$\nabla_\alpha V_\mu = \partial_\alpha V_\mu + (1+x\cdot\partial)^{-1} x^\sigma R^\nu{}_{\alpha\mu\sigma} V_\nu \quad (11b)$$

$$\begin{aligned} \nabla_\alpha V^{\mu\nu} &= \partial_\alpha V^{\mu\nu} - (1+x\cdot\partial)^{-1} x^\sigma R^\mu{}_{\alpha\lambda\sigma} V^{\lambda\nu} \\ &\quad - (1+x\cdot\partial)^{-1} x^\sigma R^\nu{}_{\alpha\lambda\sigma} V^{\mu\lambda} \end{aligned} \quad (11c)$$

$$\begin{aligned} \nabla_\alpha V_\nu^\mu &= \partial_\alpha V_\nu^\mu - (1+x\cdot\partial)^{-1} x^\sigma R^\mu{}_{\alpha\lambda\sigma} V_\nu^\lambda \\ &\quad + (1+x\cdot\partial)^{-1} x^\sigma R^\lambda{}_{\alpha\nu\sigma} V_\lambda^\mu \end{aligned} \quad (11d)$$

$$\begin{aligned} \nabla_\alpha V_{\mu\nu} &= \partial_\alpha V_{\mu\nu} + (1+x\cdot\partial)^{-1} x^\sigma R^\lambda{}_{\alpha\mu\sigma} V_{\lambda\nu} \\ &\quad + (1+x\cdot\partial)^{-1} x^\sigma R^\lambda{}_{\alpha\nu\sigma} V_{\mu\lambda} \end{aligned} \quad (11e)$$

and the Christoffel symbol-free geodesic equations,

$$\ddot{x}^k - \left\{ \begin{aligned} &(1+x\cdot\partial)^{-1} x^\sigma R^k{}_{\alpha\beta\sigma} \\ &-\dot{x}^k (1+x\cdot\partial)^{-1} x^\sigma R^0{}_{\alpha\beta\sigma} \end{aligned} \right\} \dot{x}^\alpha \dot{x}^\beta = 0. \quad (12)$$

Let us now consider the geodesic equations further. For the lowest speed approximation, that is when $\dot{x}^k = \partial x^k / \partial(ct) = v^k / c \ll 1$ the $\dot{x}^k$ -term in (10) and (12) can be neglected and accordingly the general geodesic equation (10) and the geodesic equation in the FS gauge (12) can be approximated, respectively, to

$$\ddot{x}^k \approx -\Gamma_{00}^k, \quad (13)$$

and

$$\ddot{x}^k \approx (1+x\cdot\partial)^{-1} x^\sigma R^k{}_{00\sigma}. \quad (14)$$

The right hand side of (13) and (14) are equal and are proportional to a force per unit mass. However, the Christoffel symbol in (13) and that within the Riemann tensor in (14) are in general different; the former does not fulfill the inversion formulae while the later does. Accordingly the metric corresponding to the equation (13) differs in general with that in (14).

4. FSFRW Christoffel Symbols

In the standard formalism a metric gives rise the corresponding Christoffel symbols and the Christoffel symbols result in the corresponding Riemann tensors; diagrammatically: metric $\rightarrow$ Christoffel symbol $\rightarrow$ Riemann tensor. Thus a metric becomes the source of “everything”. Given a Riemann tensor, on the other hand, we, in general, cannot derive the Christoffel symbols and the metric tensors. However, when working in the FS gauge we, by utilizing the inversion formula, are able to obtain the Christoffel symbols from the Riemann tensors. Let's say this Christoffel symbols as the Fock-Schwinger-Christoffel symbols (FSCS). In general these FSCS, obtained from certain Riemann tensors, differ from the Christoffel symbols obtained from a metric that gives the same Riemann tensors. Thus for certain Riemann tensors we have non-unique Christoffel symbols, including FSCS. For example, consider the Friedmann-Robertson-Walker (FRW) Riemann tensors:

$$\begin{aligned}
R^0_{101} = -R^0_{110} &= \frac{R\ddot{R}}{1-kr^2}, & R^0_{202} = -R^0_{220} &= r^2 R\ddot{R}, & R^0_{303} = -R^0_{330} &= R\ddot{R}r^2 \sin^2 \theta, \\
R^1_{001} = -R^1_{010} &= \frac{R\ddot{R}}{R^2}, & R^1_{212} = -R^1_{221} &= (k + \dot{R}^2)r^2, & R^1_{313} = -R^1_{331} &= (\dot{R}^2 + 2k)r^2 \sin^2 \theta, \\
R^2_{002} = -R^2_{020} &= \frac{R\ddot{R}}{R^2}, & R^2_{112} = -R^2_{121} &= -\frac{\dot{R}^2 + k}{1-kr^2}, & R^2_{323} = -R^2_{332} &= (\dot{R}^2 + k)r^2 \sin^2 \theta, \\
R^3_{003} = -R^3_{030} &= \frac{R\ddot{R}}{R^2}, & R^3_{113} = -R^3_{131} &= -\frac{\dot{R}^2 + k}{1-kr^2}, & R^3_{223} = -R^3_{232} &= -r^2(\dot{R}^2 + k)
\end{aligned} \tag{15}$$

(Other $R^\alpha_{\beta\gamma\delta}$ vanish.). There are derived from the FRW metric $g_{\alpha\beta}$ defined through the line element ds :

$$\begin{aligned}
ds^2 &= g_{\alpha\beta} dx^\alpha dx^\beta = (dx^0)^2 - R^2(t) \\
&\left[\frac{(dr)^2}{1-kr} + r^2(d\theta)^2 + r^2 \sin^2 \theta (d\varphi)^2 \right]. \tag{16}
\end{aligned}$$

$R(t)$ is a function of time and is called a cosmic scale factor. Inserting the above Riemann tensors into the inversion formulae we have the FSFRW Christoffel symbols ($\Gamma^\rho_{FS\alpha\beta}$) shown in the Appendix 1. For comparison we show in Appendix 2 the standard FRW Christoffel symbols which are derived from the above metric via the relation

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\sigma} [g_{\alpha\sigma,\beta} + g_{\beta\sigma,\alpha} - g_{\alpha\beta,\sigma}]. \tag{17}$$

It turns out that both sets of Christoffel symbols are quite different. For example, $\Gamma^\rho_{FS\alpha\beta}$ have no symmetry properties, $\Gamma^\rho_{FS\alpha\beta} \neq \Gamma^\rho_{FS\beta\alpha}$ while $\Gamma^\rho_{\alpha\beta}$ are symmetrical, $\Gamma^\rho_{\alpha\beta} = \Gamma^\rho_{\beta\alpha}$. A metric $g_{FS\mu\nu}$ having a standard relationship (17) with $\Gamma^\rho_{FS\alpha\beta}$ differs, accordingly, from the FRW metric defined in (16).

5. Conclusions

The main feature of the FS gauge is the appearance of the inversion formulae. In the general theory of relativity the formula is an expression of the Christoffel symbol in terms of Riemann tensors, a reverse expression compared to the standard relationship between the Christoffel symbol and the Riemann tensor. The formula enables one to eliminate Christoffel symbols from all expressions in general relativity resulting in Christoffel symbol-free reformulation of the general relativity theory. The formula also shows, for certain Riemann tensors, the non-uniqueness of the Christoffel symbols: there are usual Christoffel symbol and FS Christoffel symbol. The former is derived from a metric that gives rise the corresponding Riemann tensor while the later is obtained from the Riemann tensor via the inversion

formula. Explicit calculation of both kind of Christoffel symbols related to the FRW Riemann tensors have been carried out. There significantly differ.

References

1. V.A. Novikov *et al.*, *Fortschritte der Physik* **32(11)**, 585, 1984; W. Kummer, *J Weiser - Zeitschrift fur Physik C* **31**, 105, 1986; M. Schaden, S. Glazek, E. Werner, *Phys. Rev. D* **36(6)** 1922, 1987; J. Barcelos-Neto, CAP Galvao, and P. Gaete, *Mod. Phys. Lett. A* **6** 1991, 1597; R. Delbourgo and Triyanta, *Intl. J. Mod. Phys. A* **7 (23)** 5833, 1992; S. Leupold and H. Weigert, *Phys. Rev. D* **54**, 7695-7709, 1996; N. Brambilla and A. Vairo, *Phys. Rev. D* **56**, 1445-1454, 1997; P. Ball, and VM Braun, *Arxiv preprint hep-ph/9701238*, 1997; NA Sveshnikov, EG Timoshenko, *Arxiv preprint hep-th/9802161*, 1998 ; R.D. Dikeman, *Arxiv preprint hep-ph/9805401*, 1998; M. Hott and G. Metikas, *Phys. Rev. D* **60**, 067703, 1999; M. Faber *et al.*, *Phys. Rev. D* **62**, 025019, 2000; D.-W. Wang, M.-Q. Huang, and C.-Z. Li, *Phys. Rev. D* **65**, 094036, 2002; F. Bastianelli, O. Corradini, and A. Zirotti, *Phys. Rev. D* **67**, 104009, 2003.
2. W.E. Brittin, W.R. Smythe and W. Weys, *Am. J. Phys.* **50**, 693, 1982; J.R. Ellis, *J.Phys. A: Math .Gen* **23** 65-74, 1990; C.A.P. Galvao and P. Gaete, in *Physical and Non-standard Gauges*, eds. P. Gaigg, W. Kummer, and M. Schweda, Springer-Verlag, 1990.
3. L. Durand and E. Mendel, *Phys. Rev D* **26(2)**, 1368, 1982.
4. G. Modanese and M. Toller, *J.Math Phys.* **31**, 452, 1990.
5. Triyanta, *General Theory of Relativity Without Christoffel Symbols*, Proc. of Int.Conf.on Differential Equation, Kluwer 1997.
6. See, for example, M. Carmeli, *Classical Fields: General Relativity and Gauge Theory*, John Wiley & Sons 1982.

Appendix 1: Non-zero FSFRW Christoffel Symbols

$$\begin{aligned}
\Gamma_{FS01}^0(x) &= -\int_0^1 d\lambda \lambda r \frac{R(\lambda t) \ddot{R}(\lambda t)}{1 - k\lambda^2 r^2} & \Gamma_{FS11}^0(x) &= +\int_0^1 d\lambda \lambda c t \frac{R(\lambda t) \ddot{R}(\lambda t)}{1 - k\lambda^2 r^2} \\
\Gamma_{FS02}^0(x) &= -\int_0^1 d\lambda \lambda^3 \theta r^2 R(\lambda t) \ddot{R}(\lambda x) & \Gamma_{FS22}^0(x) &= +\int_0^1 d\lambda \lambda^3 c t r^2 R(\lambda t) \ddot{R}(\lambda x) \\
\Gamma_{FS03}^0(x) &= -\int_0^1 d\lambda \lambda^3 \varphi r^2 R(\lambda t) \ddot{R}(\lambda x) \sin^2(\lambda \theta) & \Gamma_{FS33}^0(x) &= +\int_0^1 d\lambda \lambda^3 c t r^2 R(\lambda t) \ddot{R}(\lambda x) \sin^2(\lambda \theta) \\
\Gamma_{FS00}^1(x) &= -\int_0^1 d\lambda \lambda r \frac{R(\lambda t) \ddot{R}(\lambda t)}{R^2(\lambda t)} & \Gamma_{FS12}^1(x) &= -\int_0^1 d\lambda \lambda^3 r^3 \theta [k + \dot{R}(\lambda t)^2] \\
\Gamma_{FS10}^1(x) &= +\int_0^1 d\lambda \lambda c t \frac{R(\lambda t) \ddot{R}(\lambda t)}{R^2(\lambda t)} & \Gamma_{FS13}^1(x) &= -\int_0^1 d\lambda \lambda^3 r^2 \varphi [\dot{R}^2(\lambda t) + 2k] \sin^2(\lambda \theta) \\
\Gamma_{FS00}^2(x) &= -\int_0^1 d\lambda \lambda \theta \frac{R(\lambda t) \ddot{R}(\lambda t)}{R^2(\lambda t)} & \Gamma_{FS21}^2(x) &= -\int_0^1 d\lambda \lambda r \frac{\dot{R}^2(\lambda t) + k}{1 - k\lambda^2 r^2} \\
\Gamma_{FS11}^2(x) &= +\int_0^1 d\lambda \lambda \theta \frac{\dot{R}^2(\lambda t) + k}{1 - k\lambda^2 r^2} & \Gamma_{FS23}^2(x) &= -\int_0^1 d\lambda \lambda^3 \varphi r^2 [\dot{R}^2(\lambda t) + k] \sin^2(\lambda \theta) \\
\Gamma_{FS20}^2(x) &= +\int_0^1 d\lambda \lambda c t \frac{R(\lambda t) \ddot{R}(\lambda t)}{R^2(\lambda t)} & \Gamma_{FS33}^2(x) &= +\int_0^1 d\lambda \lambda \theta r^2 [\dot{R}^2(\lambda t) + k] \sin^2(\lambda \theta) \\
\Gamma_{FS00}^3(x) &= -\int_0^1 d\lambda \lambda \varphi \frac{R(\lambda t) \ddot{R}(\lambda t)}{R^2(\lambda t)} & \Gamma_{FS30}^3(x) &= +\int_0^1 d\lambda \lambda c t \frac{R(\lambda t) \ddot{R}(\lambda t)}{R^2(\lambda t)} \\
\Gamma_{FS11}^3(x) &= +\int_0^1 d\lambda \lambda \varphi \frac{\dot{R}^2(\lambda t) + k}{1 - k\lambda^2 r^2} & \Gamma_{FS31}^3(x) &= -\int_0^1 d\lambda \lambda r \frac{\dot{R}^2(\lambda t) + k}{1 - k\lambda^2 r^2} \\
\Gamma_{FS22}^3(x) &= +\int_0^1 d\lambda \lambda^3 \varphi r^2 [\dot{R}^2(\lambda t) + k] & \Gamma_{FS32}^3(x) &= -\int_0^1 d\lambda \lambda \theta r^2 [\dot{R}^2(\lambda t) + k]
\end{aligned}$$

Appendix 2: Non-zero Standard FRW Christoffel symbols

$$\begin{aligned}
\Gamma_{11}^0 &= \frac{R\dot{R}}{1 - kr^2} & \Gamma_{22}^0 &= r^2 R\dot{R} & \Gamma_{33}^0 &= r^2 R\dot{R} \sin^2 \theta \\
\Gamma_{01}^1 &= \Gamma_{10}^1 = \frac{\dot{R}}{R} & \Gamma_{11}^1 &= \frac{kr}{1 - kr^2} & \Gamma_{22}^1 &= -r(1 - kr^2) & \Gamma_{33}^1 &= -r(1 - kr^2) \sin^2 \theta \\
\Gamma_{02}^2 &= \Gamma_{20}^2 = \frac{\dot{R}}{R} & \Gamma_{12}^2 &= \Gamma_{21}^2 = \frac{1}{r} & \Gamma_{33}^2 &= -\sin \theta \cos \theta \\
\Gamma_{03}^3 &= \Gamma_{30}^3 = \frac{\dot{R}}{R} & \Gamma_{13}^3 &= \Gamma_{31}^3 = \frac{1}{r} & \Gamma_{23}^3 &= \Gamma_{32}^3 = \frac{\cos \theta}{\sin \theta}
\end{aligned}$$